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Colonel Richard's Game

"Colonel Richard's Game," a two-person zero-sum game, is introduced. The game is a variant of the popular "Colonel Blotto's Game," which we will discuss first. Solutions to Colonel Richard's Game are presented and discussed.

A *game* is a mathematical model of a confrontation. Each side has a number of possible courses of action called *pure strategies*. Games are defined by their *payoff matrices*, which list the numerical outcomes for each possible choice of strategy by the different sides. In a *two-player zero-sum* (TPZS) *game*, one side's gain is the other side's loss.

A *mixed strategy* is one in which the choice of a pure strategy is probabilistic. The probability with which a side chooses each pure strategy is listed in a *strategy vector*, which defines the mixed strategy. The *rating* of a mixed strategy is the expected outcome of the game if the opponent uses the best counterstrategy. An *optimal strategy* is one that produces the best rating, and the best rating is the *value* of the game for that side. Depending on the game, a side may have more than one optimal strategy.

The Fundamental Theorem of Game Theory states that the game values for the two sides of a TPZS game add to zero.

When two strategies (one for each side) have ratings that are the negative of one another, then the strategies are necessarily optimal and the rating is the value of the game. In solving a game, we look for the two optimal strategies and the game's value.

(For a more detailed introduction to game theory, see Refs. 1 and 2.)

Colonel Blotto's Game

Colonel Blotto's Game, well known to game-theory enthusiasts, is played between two antagonists, Blue and Red (see the box "Who Was Blotto?"). Colonel Blotto, leader of the Blue forces, battles with Captain Kije, commander of the Red forces, for control of a number of passes through a mountain range that separates the two armies.

Blotto commands N units of military force, while Kije disposes of M units. Each leader allocates his or her units among the mountain passes: so many to pass 1, so many to pass 2, so many to pass 3, and so on. On the day of confrontation, control of a pass is won by the side with more units of force at that location. If the two forces at a pass are equinumerous, neither side gains control of the pass. Each side gains a point for each pass it controls and loses a point for each pass controlled by the opponent. The sum of the two sides' scores is therefore zero. Hence, Colonel Blotto's Game is TPZS.

Blotto disposes his forces to maximize his gain and Kije her forces to minimize her loss. For our purposes, the salient aspects of a Blotto game are as follows:

(1) Each commander has finite resources that are partitioned among a limited number of sites (i.e., passes).

(2) The gain resulting from the confrontation at a site depends only on the local forces. That is, conditions at one site do not influence the outcome at another.

As a simple example, Fig. 1 (top) shows a case in which there are four passes, and $N = 4$ and $M = 3$. If Blotto and Kije make their assignments as shown in Fig. 1 (center), Blue will capture two passes and Red will capture one, while control of pass 4 will be gained by neither. This result is shown in Fig. 1 (bottom). Thus, the score is +1 (i.e., 2 − 1 + 0) for Blue and, consequently, −1 for Red.

Since Blue had more forces than Red, it is not surprising that his score is positive. The game is not fair. But is a score of 1 all that Blue can expect? Is there a way to do better? If this game

Who Was Blotto?

Use of the name Blotto for the protagonist is readily traced to the paper "A Problem in Strategy" by J.W. Tukey [1]. Tukey, at least in the condensed form in which his paper appeared in *Econometrica* (vol. 17), speaks of the colonel as one with whom the reader should be familiar. Yet Tukey's paper appears to be the earliest journal reference to Blotto.

A clue is supplied by E.M.L. Beale and G.P.M. Heselden [2], who reference a problem in H. Phillips's book [3] but, oddly, misquote it. Phillips, writing as "Caliban" (Problem 26, p. 39), tells of Colonel Blotto being assigned a task described in part as

> there are four fortresses in the mountains. They are occupied by three hostile units. . . . The distribution of these units among the four fortresses is not known. . . . You are in command of four units. You are to issue orders for the immediate occupation therewith of one or more of the . . . fortresses.

The scoring is somewhat different from our case: Caliban gives Blotto credit for the number of enemy units overcome, as well as for the number of sites captured.

Because Phillips, an outstanding English propounder of mathematical and logical problems, published his book more than half a century ago, one might suspect that Blotto is overdue for retirement. The colonel, however, continues to wage war within the pages of many current books on game theory.

Interestingly enough, Blotto's antagonist was initially nameless. But S. Karlin [4] revealed in 1959 that the name is Kije (Russian for "whatchamacallit"), a captain and a dangerous opponent. Whether Karlin coined the name himself or borrowed it from Prokofiev's *Lieutenant Kije Suite* (op. 60) is not known. We shall assume that Kije is female.

References

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2. E.M.L. Beale & G.P.M. Heselden, "An Approximate Method of Solving Blotto Games," *Naval Research Logistics Quarterly*, p. 65 (1962).
3. H. Phillips, *The Week-End Problems Book* (Nonesuch Press, London, 1933).
4. S. Karlin, *Mathematical Methods and Theory in Games, Programming, and Economics*, Vol. I (Addison-Wesley, Reading, MA, 1959), pp. 82–84.

is played optimally and repeatedly, how many points, on the average, can Blue win? What constitutes optimal play? These are some of the questions that motivate game theory.

Introduction to Colonel Richard's Game

Colonel Richard's Game is a variant of Colonel Blotto's Game. The two games share the first aspect mentioned above (that of finite, partitioned resources), but they differ in the second aspect (i.e., conditions at one site *can* influence the outcome at another).

In its full ramifications, Colonel Richard's Game is more difficult to solve. This article deals with an elementary version of the game. We give some results for the elementary game (both for their intrinsic interest and to introduce the new game to the reader) in anticipation of future results of more complex versions of the game.

This article presents exact solutions for the cases of equal forces ($M = N$) and inferior defenses ($M < N$), and comments on the character of those solutions. We also derive solutions for the case in which Red has a large preponderance of defense weapons ($M > \sim 1.4N$). Furthermore, solutions for $M > N$ for small values of $M - N$ are exhibited. We have not yet, however, derived a general solution for this case.

Rules

Two forces are engaged in battle: Colonel Rich-

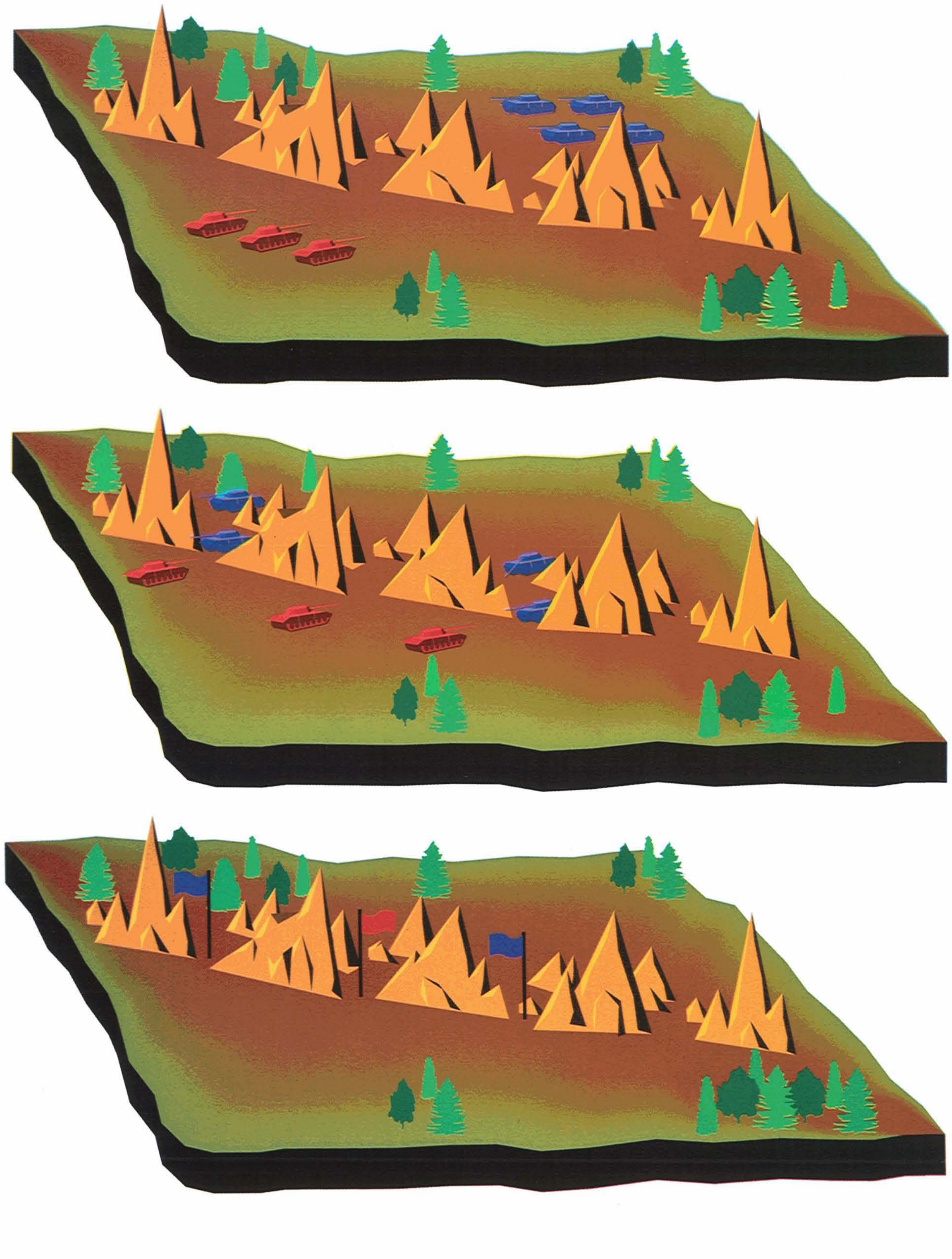


Fig. 1—Colonel Blotto's Game for N = *4 and* M = *3.*

ard, leader of Blue, fires missiles at a base that is defended by Captain Kije, commander of Red. The base, which is located near a coast, is protected by a radar that can detect Blue's missiles and guide interceptors to destroy them. Blue, however, has a radar-attack boat (RAB) from which missiles can be fired at the radar in an attempt to knock it out. Red can try to protect the sensor by allocating interceptors to neutralize Blue's missiles. Following the struggle over the radar, Colonel Richard fires his remaining missiles from a base-attack boat (BAB) at the Red base itself, and the missiles are countered by the remaining Red interceptors.

The two phases of the battle are governed by the following rules.

Phase 1: The radar site. Some (or none) of the Blue missiles are launched against the radar, which is defended by some (or none) of the Red interceptors. If the number of attacking missiles exceed the number of defending interceptors, the radar is destroyed; otherwise it is unharmed. Figure 2 (top) shows the situation before the engagement begins, and Fig. 2 (center) the attack on the radar.

Phase 2: The base site. The remaining Blue missiles are then launched against the base, which is defended by the remaining Red interceptors. If the radar was knocked out in phase 1, all missiles reach their target. If the radar is still operational, each Red interceptor neutralizes one Blue missile, and only the excess missiles, if any, reach their target. Figure 2 (bottom) shows the attack on the base.

The following rules apply to the battle as a whole.

Allocation. Colonel Richard can partition his missiles between the radar and base in any way, but the missiles on the RAB cannot be used against the base, nor those on the BAB against the radar. Similarly, after Captain Kije has assigned certain interceptors to protect the radar, she cannot use them to protect the base, nor can the base interceptors protect the radar.

Scoring. Blue scores one point for each missile that reaches its target at the base. Destroying the radar does not, per se, gain points for Blue.

Foreknowledge. Each side knows the other's stockpile of weapons, but not the allocation of weapons between the two sites.

In a generalization of Colonel Richard's Game, Blue may have several boats of both types and Red may have several radars. However, we will consider only the elementary version of the game in which Blue has one RAB and one BAB, and Red one radar. We concentrate on the case in which Blue and Red have the same number of weapons, then consider the case of different numbers of weapons.

Payoff Matrix of the Elementary Game, Equal Stockpiles

If Blue and Red have N weapons each, the elementary game has a payoff matrix $\mathbf{G}(N)$ of the form

$$\mathbf{G}(N) = \begin{bmatrix} 0 & 1 & 2 & 3 & \ldots & N \\ N-1 & 0 & 1 & 2 & \ldots & N-1 \\ N-2 & N-2 & 0 & 1 & \ldots & N-2 \\ \ldots & & & & & \\ 1 & 1 & 1 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}.$$

The entry in the ith row and jth column of $\mathbf{G}(N)$, $g_{i,j}$, is the gain to Blue (equal to the loss to Red) if Blue fires i missiles at the radar and Red defends the sensor with j interceptors ($0 \le i, j \le N$). (Note: For the sake of convenience, we enumerate the rows and columns of a payoff matrix as starting from the zeroth, not the first, row and column. The ith row, then, corresponds to i missiles attacking the radar, and the jth column to the allocation of j interceptors to defend the sensor. A similar convention applies to vectors.) Thus:

(*a*) The entries on the main diagonal, $g_{i,i} = 0$, correspond to the outcome in which the i missiles attacking the radar in phase 1 are successfully countered by the i interceptors defending it, and the $N - i$ missiles attacking the base in phase 2 are neutralized (with the assistance of the radar) by the $N - i$ interceptors defending the base. Blue's gain is zero, because no missiles reach the base.

(*b*) The entries above the main diagonal

$$g_{i,j} = j - i, \; j > i$$

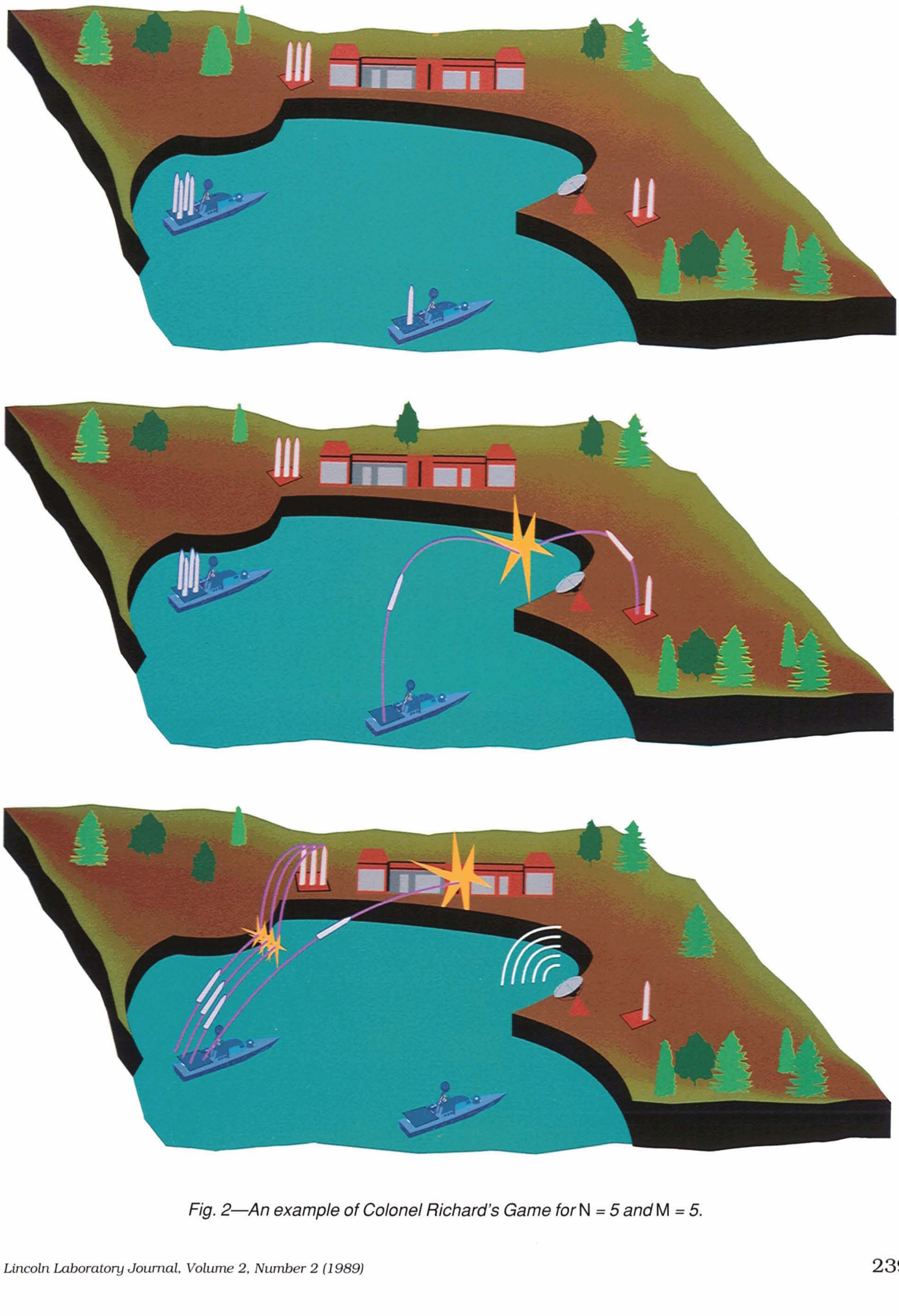


Fig. 2—An example of Colonel Richard's Game for N = *5 and* M = *5.*

correspond to the outcome in which Red allocates more interceptors to defend the radar than Blue sends missiles to attack it. Although the radar survives to guide the base interceptors, there are fewer of these interceptors than here are attacking missiles, and the excess missiles score. In the extreme case in which Red uses every interceptor to defend the radar ($j = N$), every missile that Blue fires against the base scores.

(c) The entries below the main diagonal

$$g_{i,j} = N - i, \; j < i$$

correspond to the outcome in which the radar is destroyed because too few interceptors were defending it. Hence, all the missiles launched against the base in phase 2 score.

From both the description of the game and the elements of $\mathbf{G}(N)$, it is clear that if Red knew Blue's *play* (i.e., his choice of radar-attack missiles, represented by the value of i), she could prevent him from scoring by choosing $j = i$. In other words, Red would use as many interceptors to defend the radar as there are attacking missiles. Conversely, if Blue knew Red's *play* (i.e., her choice of radar-defense interceptors, represented by the value of j), he would attack the radar with just the number of missiles necessary to gain the maximum possible score. Depending on the value of j, Blue would attack the radar with either 0 or $j + 1$ missiles. (Note that Blue would not necessarily choose to destroy the radar.) However, under our assumptions neither side knows the other's choice.

As we shall see, except for the trivial case of $N = 1$, there is no saddle point in the game, and each party should use a mixed strategy. (For a definition of saddle point, see the box "Saddle Points.") That is, Blue should choose the value of i from some probability distribution that depends on N; similarly, Red should choose the value of j according to some other stochastic law. If both parties play properly, the resulting value of the game, υ, is optimal in the usual minimax sense of game theory. That is, υ is as large as Blue could expect in repeated play against an intelligent Red player and, at the same time, υ is as small as Red can expect against an intelligent Blue player. By the nature of the game, of course, υ is a non-negative number, $0 \le \upsilon \le N$.

The inability of either commander to predict the opponent's exact strategy—because of the probabilistic way in which strategies are chosen—is central to the play of the game.

The solution of Colonel Richard's Game takes the form of two *strategy vectors* **B** and **R**, each containing $N + 1$ elements. The ith element of **B**, b_i, is the probability that Blue allocates i missiles to attack the radar and the jth element of **R**, r_j, is the probability that Red allocates j interceptors to defend the sensor. Here, three conditions apply: $0 \le i, j \le N$; each element of **B** and **R** must lie in the interval 0 to 1; and the sum of the elements of each vector must be unity.

It is important to note that a player's available strategies need not all be represented by nonzero probabilities in the strategy vector. Some pure strategies (i.e., choices of i or j) are so poor that they should never be played. Such strategies are known as *dominated* strategies. Strategies that are played with nonzero probabilities are known as *active* strategies. (See the box "Dominance.")

Solution to Colonel Richard's Game

Let **J** be a row vector consisting of $N + 1$ elements that are all equal to 1. If the payoff matrix is such that all strategies are active, we can express the value of the game and the two optimal strategies as

$$\upsilon = \frac{1}{\mathbf{J}\mathbf{G}^{-1}\mathbf{J}^{\mathrm{T}}} \tag{1}$$

$$\mathbf{B} = \upsilon \mathbf{J}\mathbf{G}^{-1} \tag{2}$$

$$\mathbf{R} = \upsilon \mathbf{J}\,(\mathbf{G}^{-1})^{\mathrm{T}}\,. \tag{3}$$

(See chapter 2 in Ref. 1, chapter 2 in Ref. 2, or chapter 3 in Ref. 3.) Equations 1 through 3 can be modified to account for the case in which **G** is singular. However, all the matrices we wish to invert will, in fact, be regular. Note that in this

Saddle Points

Payoff matrices can contain one or more *saddle points* (SP). An SP is a matrix element that is both the minimum value in its row and the maximum value in its column. If an SP exists, the optimal play by Blue is to choose the row containing the SP, and Red's optimal play is to choose the column containing the SP. The value of the game is then the value of the SP. Consequently, the existence of an SP results in both players adopting pure strategies; i.e., it does not matter that Red knows exactly what Blue will do because she cannot take advantage of the information. Nor can Blue take advantage of knowing Red's pure strategy.

If a payoff matrix contains more than one SP, it can be shown that the multiple SPs all have the same value. Therefore, it doesn't matter which one is played.

article, **G** will sometimes be used in place of **G**(N).

We can also determine the mean number of missiles that Blue will launch at the radar; it is

$$\bar{b} = \mathbf{BK}^{\mathrm{T}} \tag{4}$$

where **K** is an (N + 1)-element row vector whose ith entry is $k_i = i$.

Similarly, the mean number of interceptors that Red will assign to protect the radar is

$$\bar{r} = \mathbf{RK}^{\mathrm{T}}\,. \tag{5}$$

Since the game **G**(N) has value v, it is important to note that for Blue to attack the base with fewer than v missiles is a dominated (i.e., inferior) strategy. And, since the radar will therefore never be attacked by more than $N - v$ missiles, for Red to defend the sensor with more than $N - v$ interceptors is also a dominated strategy. It is convenient at this point to introduce the quantity

$$n = [N - v] \tag{6}$$

where [. . .] is the greatest-integer-not-exceeding function. Thus, Blue fires up to n missiles at the radar and Red defends the sensor with as many as n weapons. Consequently, each side has (n + 1) active strategies.

Results for Small N

We examine the game in detail for small values of N.

For the trivial case of $N = 1$, the payoff matrix is

$$\mathbf{G}(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Blue scores only if he sends his single missile to the base and Red uses her single interceptor to defend the radar. Uniquely, this game has a saddle point: $g_{0,0}$; i.e., Red will choose $j = 0$ and the value of the game will be $v = 0$, regardless of what Blue does.

For the case of $N = 2$, the payoff matrix is

$$\mathbf{G}(2) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The third column is dominated by each of the other columns, and the third row is dominated by each of the other rows. These conditions imply, respectively, that strategies $j = 2$ and $i = 2$ are inactive. By discarding them, we end up with a *reduced matrix* (denoted by **G'**), which contains only active strategies,

$$\mathbf{G}'(2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

whence, from Eqs. 1 through 5,

$$\mathbf{B} = [1/2 \quad 1/2 \quad 0]$$

$$\mathbf{R} = [1/2 \quad 1/2 \quad 0]$$

$$\text{and} \quad v = 1/2\,; \;\; \bar{b} = \bar{r} = 1/2\,.$$

Dominance

Dominance of one strategy over another plays an important role in game theory. Consider the game shown in Fig. 1. It is clear that strategy i_1 for Blue is always superior to strategy i_2, regardless of Red's move: i.e., there is never a reason to play i_2. We say that strategy i_1 *dominates* strategy i_2. Similarly, Fig. 2 shows an example in which Red's strategy j_1 dominates the alternative strategy j_2.

A *dominated* strategy should be played with zero probability; consequently, all dominated strategies may be dropped from consideration. Eliminating dominated strategies reduces the size of the payoff matrix.

Note that it is possible to have a condition of *partial dominance*, as exemplified in Fig. 3. Here, Blue can do at least as well with strategy i_1 as with strategy i_2. Thus the latter strategy is unnecessary. For our purposes, we will discard partially dominated strategies as well as dominated ones; i.e., we will make no distinction between dominated and partially dominated strategies.

$$\begin{array}{c} \\ i_1 \\ \\ i_2 \\ \\ \end{array}\begin{bmatrix} \cdots & & & & & \\ 5 & 5 & 8 & 9 & 4 & 3 \\ \cdots & & & & & \\ 4 & 3 & 7 & 6 & 1 & 0 \\ \cdots & & & & & \end{bmatrix}$$

(1)

$$\begin{array}{ccccc} & j_1 & & j_2 & \end{array}$$
$$\begin{bmatrix} \cdots & 2 & \cdots & 9 & \cdots \\ & 1 & & 2 & \\ & 1 & & 8 & \\ & 0 & & 6 & \\ & 6 & & 7 & \end{bmatrix}$$

(2)

$$\begin{array}{c} \\ i_1 \\ \\ i_2 \\ \\ \end{array}\begin{bmatrix} \cdots & & & & & \\ 4 & 4 & 8 & 9 & 4 & 0 \\ \cdots & & & & & \\ 4 & 3 & 7 & 6 & 4 & 0 \\ \cdots & & & & & \end{bmatrix}$$

(3)

Note that, in forming **B** and **R,** we insert zeros in the locations of the inactive strategies, which are not represented in **G'**.

For the case of $N = 3$, the payoff matrix is

$$\mathbf{G}(3) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Column 3 and row 3 are dominated. Thus, the matrix reduces to

$$\mathbf{G}'(3) = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We find

$$\mathbf{B} = [1/3 \quad 1/6 \quad 1/2 \quad 0]$$

$$\mathbf{R} = [1/3 \quad 1/2 \quad 1/6 \quad 0]$$

$$\text{and} \quad \upsilon = 5/6\,; \quad \bar{r} = 5/6\,; \quad \bar{b} = 7/6\,.$$

Note that $\bar{r} = \upsilon$, as was the case for $N = 2$. It can be easily proven [4] that $\bar{r} = \upsilon$ for all $\mathbf{G}(N)$.

Solution for Any N

The reduced game matrix $\mathbf{G}'(N)$ is square and has $n + 1$ active strategies for each player (n was defined in Eq. 6).

Thus, the reduced matrix can be written as

$$\mathbf{G}' = \begin{bmatrix} 0 & 1 & 2 & \cdots & n-1 & n \\ N-1 & 0 & 1 & \cdots & n-2 & n-1 \\ N-2 & N-2 & 0 & \cdots & n-3 & n-2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ N-n+1 & \cdot & \cdot & \cdots & 0 & 1 \\ N-n & N-n & N-n & \cdots & N-n & 0 \end{bmatrix}$$

We observe that

$$\mathbf{BG}' = \upsilon \mathbf{J}\,.$$

(Recall that $\mathbf{J}$ is the $(n + 1)$-element row vector in which every entry is a 1.) An application of difference equations gives us a general solution for the Blue strategy vector **B:**.

$$b_i = \frac{(N-n)}{(N-i)\,(N-i+1)}\,, \quad i = 1, 2, \ldots, n$$

$$b_0 = 1 - n/N\,.$$

For the Red strategy vector **R** we have the equation

$$\mathbf{G}'\,\mathbf{R}^{\mathrm{T}} = v\mathbf{J}^{\mathrm{T}} .$$

Hence,

$$r_i = \frac{1}{(N-i)} , \quad i = 0, 1, 2, \ldots, n-1$$

$$r_n = 1 - \frac{v}{(N-n)} .$$

The value of the game is given by

$$v = (N-n) \sum_{k=0}^{n-1} \frac{1}{(N-k)} .$$

When we consider the limiting case in which $N \rightarrow \infty$, we find an interesting solution. The value of the game converges to $v = N/e$, and the size of the maximum attack on the radar converges to $n = N(1 - 1/e)$.

The two optimal strategies **B** and **R** can be represented as continuous probability density functions (PDF) when their units are scaled down by the factor N. Let x be that scaled unit for Blue; thus, b_x is the probability with which Blue should use Nx missiles to attack the radar. Then

$$b_0 = 1/e$$

$$b_x = \left(\frac{1}{N}\right) \frac{1/e}{(1-x)^2} , \quad 0 < x \le (1-1/e)$$

$$\bar{x} = 1 - 2/e$$

where $\bar{x}$ is the expected fraction of missiles fired at the radar. With y, r_y, and $\bar{y}$ defined analogously for Red, we find

$$r_y = \left(\frac{1}{N}\right) \frac{1}{(1-y)} , \quad 0 \le y \le (1-1/e)$$

$$\bar{y} = 1/e .$$

The top and bottom of Fig. 3 respectively show the optimal PDFs b_x and r_y.

Note that the limiting case in which $N \rightarrow \infty$ is equivalent to the continuous variation of the game in which the two sides are not restricted to using an integer number of weapons. Consequently, the above solutions can also be derived by using the differential equations of the continuous case.

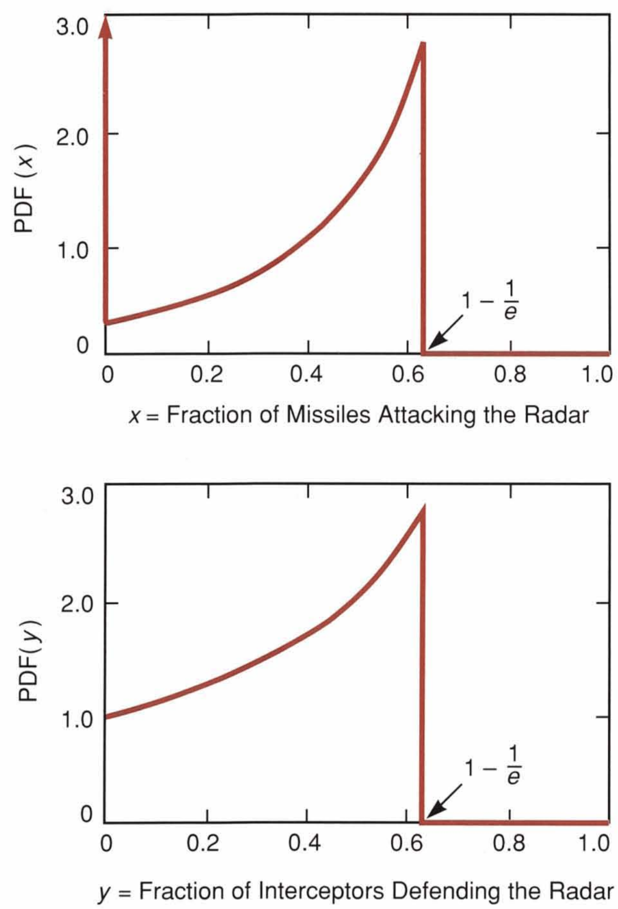


Fig. 3—(top) Blue's optimal strategy. (bottom) Red's optimal strategy.

Unequal Stockpiles

So far, we have considered the case in which Blue and Red have the same number of weapons. We now extend our discussion to the case of unequal stockpiles. We use the symbol $\mathbf{G}(N, M)$ to designate the game in which Blue has N missiles and Red has M interceptors. $\mathbf{G}(N, M)$ also identifies the payoff matrix of the game. The equinumerous game treated previously is $\mathbf{G}(N, N) = \mathbf{G}(N)$.

Red Has Fewer Weapons

For the case in which Red has fewer weapons than Blue ($M < N$), the payoff matrix for $\mathbf{G}(N, M)$ is of size $(N+1) \times (M+1)$. However, the additional $(N - M)$ rows are dominated and can be eliminated. The resulting $(M + 1) \times (M + 1)$ reduced matrix is identical to $\mathbf{G}(M)$, the matrix for the equinumerous case, except that each entry has been increased by $(N-M)$. It is easy to show (e.g., exercise in chapter 1 of Ref. 2) that the strategy vectors for $\mathbf{G}(N, M)$ are the same as those of $\mathbf{G}(M)$ and that the value of the new game is $(N - M)$ more than that of the equinumerous game. This argument disposes of the case $M < N$.

Red Has More Weapons

The situation in which Red has $\delta = M - N$ more weapons than Blue is more complicated. The form of the payoff matrix changes radically, and no simple relation between the new game and the equinumerous game is known. For example, if Blue has three missiles and Red has four interceptors, the matrix is

$$\mathbf{G}(3, 4) = \begin{bmatrix} 0 & 0 & 1 & 2 & 3 \\ 2 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

From considerations of dominance, strategies $i = 1, 3$ and $j = 0, 3, 4$ can be eliminated, which results in the reduced matrix

$$\mathbf{G}'(3, 4) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Coincidentally, we have already encountered this matrix in the game $\mathbf{G}(2)$ in an entirely different context. The strategy vectors are

$$\mathbf{B} = [1/2 \quad 0 \quad 1/2 \quad 0]$$

$$\mathbf{R} = [\,0 \quad 1/2 \quad 1/2 \quad 0 \quad 0]$$

$$\text{and} \quad v = 1/2\,; \quad \bar{b} = 1\,; \quad \bar{r} = 3/2\,.$$

This is a solution that we have not seen before. Note that $\bar{r}$ no longer equals v. Indeed, it can be shown that $\bar{r} = v + \delta$ for all values of N and M for the game $\mathbf{G}(N, M)$.

A simple argument can help us find the dominated strategies in the matrix. As before, the use of more than $n = [N - v]$ weapons either to attack or defend the radar is a dominated

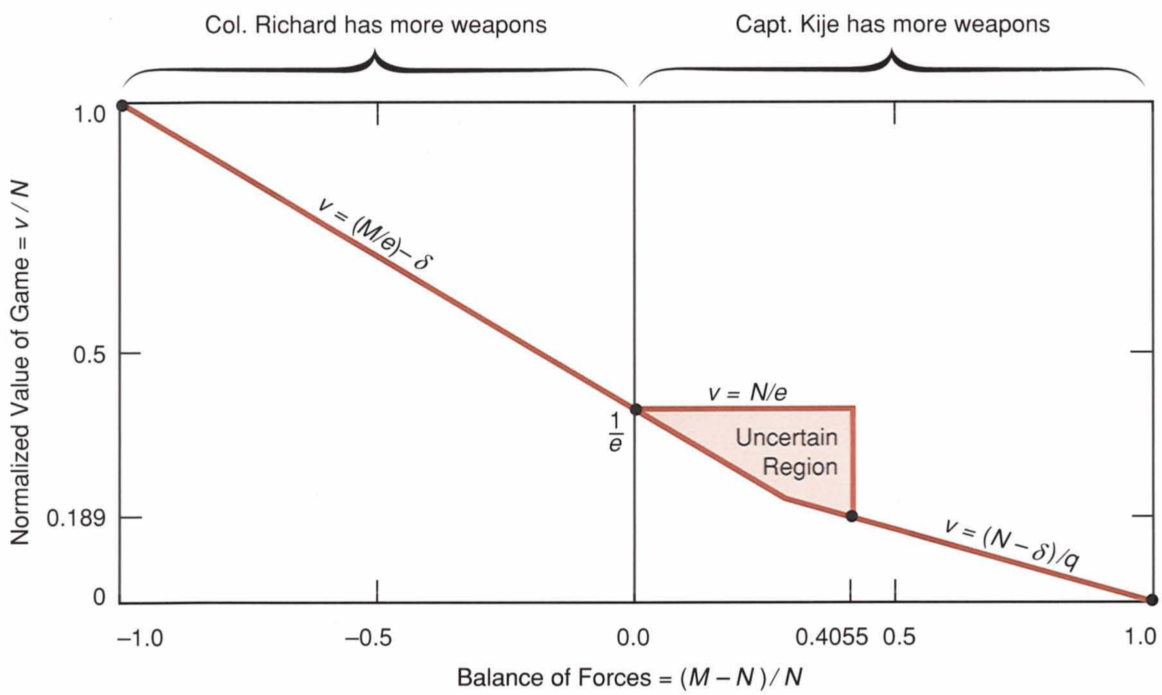


Fig. 4—Value of Colonel Richard's Game as a function of balance of forces.

strategy. Furthermore, since Red has no need for more than N interceptors to defend her base, at least δ will always be assigned to the radar. In light of this, Blue cannot hope to destroy the radar by using fewer than $(\delta + 1)$ missiles. If Blue chooses *not* to destroy the radar, he should not use even one missile against it. Therefore, the active strategies for Red are contained among

$$j = \delta,\ \delta + 1,\ \ldots,\ n;$$

and for Blue, among

$$i = 0,\ \delta + 1,\ \delta + 2,\ \ldots,\ n.$$

Numerical tests indicate (but we have not proven) that all of these strategies are in fact active.

For the case in which the preponderance of Red weapons exceeds a threshold, if Blue attempts to knock out the radar and fails to do so, Blue's remaining weapons are insufficient to overcome the base's defenses. This condition occurs when $\delta > n/2$. We have solved this case analytically for the continuous limit and found some points of interest. The case can also be solved for the usual discrete version, but the results are not illuminating. We state the results without proof.

It appears that an intriguing constant q plays an important role in the solution. The constant is defined by the equation

$$q = 2 + \ln(q),$$

which gives $q \cong 3.1461\ 9322\ 062$. (The use of q as an approximation to π is not recommended.) The threshold for the preponderance condition is given by

$$M > \frac{(3q - 2)N}{(2q - 1)}$$

$$> \sim 1.4055\ N\ .$$

It can be shown that the value of the game v is equal to $(N - \delta)/q$, and the optimal strategies for Blue, in which x is again the scaled number of missiles defined earlier, are

$$b_0 = \frac{1}{q - 1}$$

$$b_x = \left(\frac{1}{N}\right) \frac{1}{(q-1)(1-x)}\ , \quad \delta/N < x \le 1 - v/N$$

$$\bar{x} = \left|1 - \frac{1}{q-1} - \frac{1}{q}\right| + \frac{\delta/N}{q}$$

$$\cong 0.2162 + 0.3178\ (\delta/N)\ .$$

For Red, the factors are

$$r_{\delta/N} = 1/q$$

$$r_y = \left(\frac{1}{N}\right) \frac{v/N}{(1-y)^2}\ , \quad (1+\delta)/N < y \le 1 - v/N$$

$$\bar{y} = (v + \delta)/N\ ,$$

in which $\bar{y}$ is the scaled number of interceptors. Note that $\bar{y}$ agrees with r (after scaling).

Red Has Marginally More Weapons

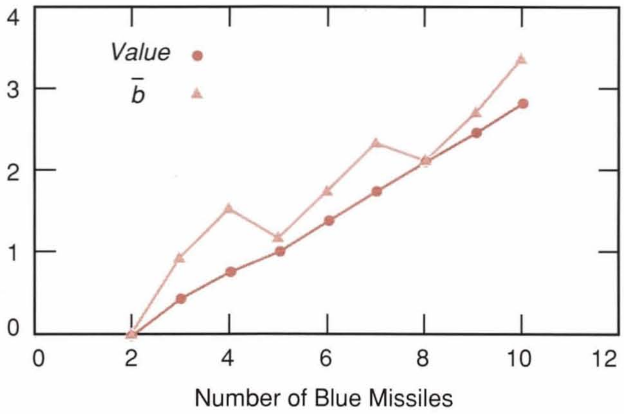
For the case in which Red has marginally more weapons (i.e., δ is small), it can be shown that by following the strategy outlined for $N > M$, Blue ensures himself a payoff of at least $M/e - \delta$. Similarly, by following the strategy outlined for $M > \sim 1.4N$, Blue will ensure himself a payoff of at least $(N - \delta)/q$. Using these observations and an argument of monotonicity, we can place a bound (shaded region in Fig. 4) on the value of the game for the case of small δ. An analytic solution has not yet been found for this case.

For small values of N, Figs. 5(a), 5(b), and 5(c) respectively show v and b for $\mathbf{G}(N, N+1)$, $\mathbf{G}(N, N+2)$, and $\mathbf{G}(N, N+3)$.

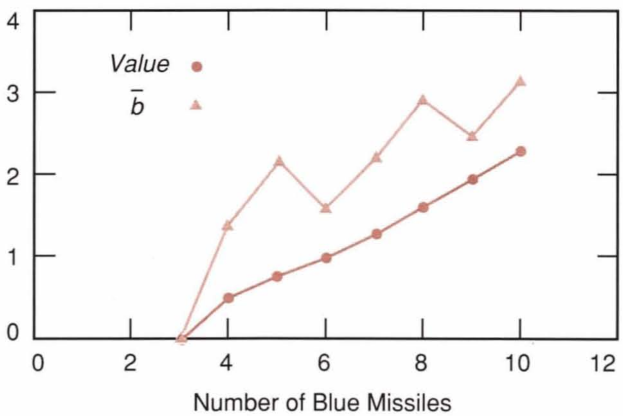
Conclusion

We have defined Colonel Richard's Game and analyzed an elementary version. Exact solutions of the equinumerous game and particular solutions of the unequal game were given.

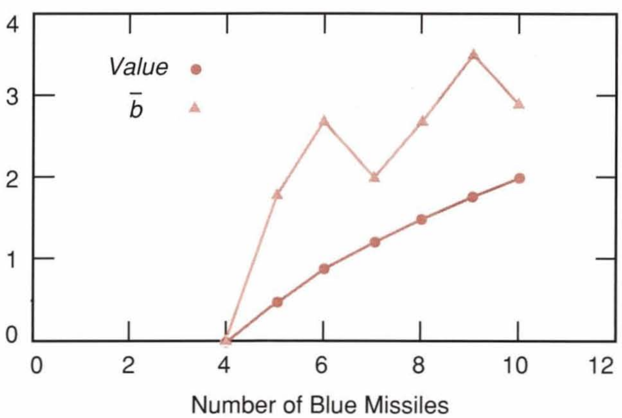
We will direct further work at cases in which Blue has multiple boats to attack the radar



(a)



(b)



(c)

Fig. 5—Value of Colonel Richard's Game and b *for small* δ *and small* N*: (a)* **G**(N, N + 1)*, (b)* **G**(N, N + 2)*, and (c)* **G**(N, N + 3).

and multiple boats to attack the base, and in which Red has more than one radar to defend its territory.

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