M. P. Shatz and G. H. Polychronopoulos

An Algorithm for the Evaluation of Radar Propagation in the Spherical Earth Diffraction Region

We develop an efficient method for computing the Airy function and apply this method to the problem of calculating the radar propagation factor for diffraction around a smooth sphere. Even for high-altitude antennas, our calculations give accurate results well into the visible region, and thus extend the useful range of the Spherical Earth with Knife Edges (SEKE) radar propagation model.

The design and analysis of radar systems that transmit over oceans or smooth terrain — for example, airborne radars that spot distant incoming missiles, or coastal radars that find drug smugglers — requires accurate evaluation of radar propagation over a smooth sphere. Analysis of smooth sphere diffraction is equally important for general radar applications; by combining several approximations, composite models can evaluate signal propagation over any type of terrain. The Spherical Earth with Knife Edges (SEKE) model [1], for instance, combines four approximations in its analysis of radar propagation over irregular terrain. (The box, "SEKE," describes this radar-propagation model in greater detail.)

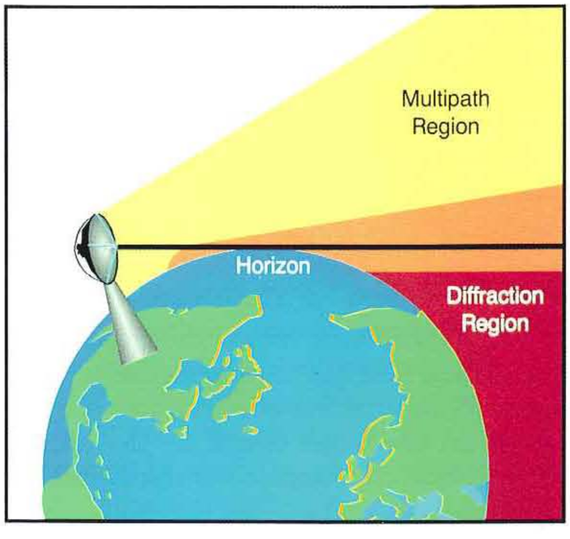

Fig. 1—The algorithm described in this paper extends the area of validity of the diffraction calculaton into the multipath region (not to scale).

Figure 1 shows three regions for propagation over a sphere: the multipath region (in yellow), well above the horizon; the diffraction region (in red) below the horizon; and an intermediate region (in orange). In the multipath region a calculation based on interference between the direct and reflected rays can be done with accuracy. In the diffraction region a small number of leading terms of an infinite series solution involving Airy functions can be used to approximate the propagation factor. In the intermediate region this series can also be used but more terms and high computational accuracy for the Airy Function are required.

When SEKE was originally developed, it was designed for low altitudes, and its method of calculating the Airy functions was sufficiently accurate. However, for high-altitude antennas or targets greater accuracy is necessary. The new method that is described in this paper provides the accuracy required for high-altitude applications for all reasonable target and antenna heights into the visible (multipath) region.

Spherical Earth Diffraction

An accurate model of diffraction about a smooth spherical earth must account for refraction in air. The index of refraction in air, n, is

SEKE

The SEKE propagation model is used to compute the field strengths of radio waves propagating from ground-based radar to low-flying aircraft, over irregular terrain. The model was developed at Lincoln Laboratory by Dr. Serpil Ayasli.

The SEKE model combines several propagation models: smooth sphere diffraction, multiple knife-edge diffraction, reflection from a smooth sphere, and a ray-tracing computation of reflections off an irregular surface. The model determines, depending on the terrain, whether to use a diffraction calculation, a reflection calculation, or a weighted average of the two methods. If a diffraction calculation is used, the model chooses the knife edge, the smooth sphere, or a weighted average of the two methods. Again, the decision depends upon the terrain. The reflection models are implicit in the two versions of SEKE: SEKE 1 uses the smooth sphere assumption; SEKE 2 uses the ray-tracing method.

The multiple knife-edge diffraction approach starts by finding the knife edges in the terrain data that most block the direct ray. The multiple knife-edge approach then implements Deygout's method [1] for computing the propagation factor.

For smooth sphere diffraction, an effective sphere is fitted to the terrain between the target and the antenna. The strength of the field is then determined by the Airy-function method described in this paper.

Smooth sphere reflection, like smooth sphere diffraction, requires fitting an effective sphere to the terrain between the target and the antenna. The specular point is then found and a new effective sphere is fitted to the first Fresnel zone. (The first Fresnel zone is the region on the ground where the length of the path from the radar to the ground and then to the target is within one-half a radar wavelength of the length of the path from the radar to the specular point and then to the target.) Finally, a third effective sphere is fitted to the first Fresnel zone of the specular point on the second sphere, and the propagation factor is computed.

The ray-tracing program finds those points on the terrain that give specular reflections. It computes the amplitude and phase of each specular point using the unshadowed points in the first Fresnel zone.

The SEKE software (written in FORTRAN) can be obtained from MIT Technology Licensing Office, E32-300, 77 Massachusetts Ave., Cambridge, MA 02139.

1. J. Deygout, "Multiple Knife-Edge Diffraction of Microwaves," *IEEE Trans. Antennas Propag.* **AP-34,** 480 (1966).

approximately given by

$$n = 1 + 7.76 \cdot 10^{-5}\frac{P}{T} + \frac{0.37e}{T^2},$$

where P is the pressure in mbar, T the temperature in K, and e the partial pressure of water in mbar. Usually n decreases with increasing altitude, which causes radio waves to bend toward the earth. A standard assumption [2], which works for normal conditions, is that the rate of change in n with height is a constant $C = -3.9 \cdot 10^{-8}\text{m}^{-1}$. Using this assumption, refraction can be accounted for by treating the radio waves as though they were in a vacuum above a sphere of radius R_{eff}, given by

$$R_{eff} = \frac{R_{earth}}{1 + C\,R_{earth}} \approx \frac{4}{3}R_{earth}.$$

This approximation — that diffraction around the earth with its atmosphere is equivalent to diffraction over a smooth sphere of radius R_{eff} in a vacuum — is used throughout this paper.

The propagation factor, F, is defined as

$$F \equiv \left|\frac{E_t}{E_0}\right| ,$$

where E_t is the electric field at the target and E_0 is the electric field at the range of the target on the axis of the antenna beam (in free space). F depends on the target height h_t, the antenna height h_a, and the range r, via the normalized parameters x, y, and z

$$x = \frac{r}{r_0},\ y = \frac{h_a}{h_0},\ z = \frac{h_t}{h_0},$$

where h_0, the normalization factor for height is

$$h_0 \equiv \frac{1}{2}\left(\frac{a\lambda^2}{\pi^2}\right)^{1/3},$$

and r_0, the normalization factor for range is given by

$$r_0 \equiv \left(\frac{a^2\lambda}{\pi}\right)^{1/3}.$$

Here λ is the wavelength of the electromagnetic wave, and a is the effective radius of the sphere.

The general solution to the calculation of the propagation factor F for a spherical earth can be expressed as an infinite series that contains Airy functions of a complex argument [3] and is dependent on x, y, and z

$$F(x,y,z) = 2\sqrt{\pi x}\sum_{n=1}^{\infty} f_n(y) f_n(z) \exp\left[\frac{1}{2}(\sqrt{3}+i)a_n x\right],$$

$$\text{where } f_n(u) \equiv \frac{Ai(a_n + ue^{\pi i/3})}{e^{\pi i/3}\,Ai'(a_n)}, \qquad (1)$$

where $Ai'(w)$ is the first derivative of the Airy function and a_n is the nth zero of the Airy function, ($a_n < 0$). The Airy function (see the box, "Properties of the Airy Function"), $Ai(w)$, is defined as

$$Ai(w) = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{t^3}{3} + wt\right) dt.$$

The series in Eq. 1 converges for all (positive) values of x, y, and z. However, the series is not numerically useful in most of the multipath region (r much less than the distance to the horizon for a specified h_t and h_a), because, in that region, the largest terms of the series are very much larger than the sum of the series. Fortunately, other approximations are valid in that region [3]. With an accurate computation of the Airy function, this series provides numerically useful results far enough into the multipath region to meet the needs of models that match diffraction and multipath calculations, *eg*, SEKE [1]. Although algorithms for the evaluation of $Ai(w)$ for real w are readily available [4], modifying them to handle complex arguments is not straightforward.

Algorithm for Evaluating the Complex Airy Function

The Airy function is entire (no singularities in the finite complex plane), so it can be expressed with a convergent power series representation [Ref. 5, Eq. 10.4.2]. This series converges very fast for small z. When a large value of z is used, the series converges very slowly, and inaccuracies due to large cancellations occur. The power series expansion for $Ai(z)$ is

$$Ai(z) = \alpha \cdot h(z) - \beta \cdot g(z), \qquad (2)$$

where α and β are constants

$$\alpha = Ai(0) + \frac{3^{-2/3}}{\Gamma(2/3)} = 0.3550280538,$$

$$\beta = -Ai'(0) = \frac{3^{-1/3}}{\Gamma(1/3)} = 0.2588194037,$$

and

$$h(z) = \sum_{k=0}^{\infty} 3^k\left(\frac{1}{3}\right)_k \frac{z^{3k}}{(3k)!} = 1 + \frac{1}{3!}z^3 + \frac{1\cdot 4}{6!}z^6 + \frac{1\cdot 4\cdot 7}{9!}z^9 + \ldots,$$

$$g(z) = \sum_{k=0}^{\infty} 3^k\left(\frac{2}{3}\right)_k \frac{z^{3k+1}}{(3k+1)!} = z + \frac{2}{4!}z^4 + \frac{2\cdot 5}{7!}z^7 + \frac{2\cdot 5\cdot 8}{10!}z^{10} + \ldots.$$

The above expressions for the power series show that each sum grows exponentially on the real axis as z becomes larger, but the difference decreases exponentially as z grows larger. Thus for large z the power series becomes numerically unstable (very sensitive to computer round-off errors).

Schulten *et al.* [6] give a method for the computation of the complex Airy function for large z. This method uses an integral representation for the Airy function that can be evaluated by a Gaussian quadrature using only a few terms. The integral representation for $Ai(z)$ is derived from an expression for the modified Hankel function $K_\nu(z)$ [Ref. 7, Eq. 6.627]

Properties of the Airy Function

The Airy function, first introduced by Airy [1], is used in solving diffraction problems, and also in mathematically analogous applications. Airy was, incidentally, known equally well for his brilliance and his arrogance. For a brief biography of this extraordinary mathematician, see the box, "Sir George Biddell Airy (1801-1892)."

The Airy function is defined by

$$Ai(z) = \frac{1}{\pi}\int_0^\infty \cos(t^3/3 + zt)\ dt.$$

The Airy function satisfies the differential equation

$$\frac{d^2Ai(z)}{dz^2} = z\,Ai(z)$$

and most of its important properties can be derived from this equation.

The differential equation has no singular points in the finite z plane and an irregular singular point at infinity. The Airy function is thus regular in the finite z plane. The possible asymptotic behaviors of the solutions to the differential equation are

$$Ai(z) \sim c\,\frac{e^{\pm 2/3\, z^{3/2}}}{\sqrt[4]{z}}.$$

The Airy integral vanishes exponentially as z increases on the positive real axis, so

$$Ai(z) \sim c\,\frac{e^{-2/3\, z^{3/2}}}{\sqrt[4]{z}}.$$

It can be shown that the constant, c, must be $\frac{1}{2\sqrt{\pi}}$. The Airy function, $Ai(z)$, decreases exponentially as $|z| \to \infty$ if $|\arg z| < \pi/3$ and increases exponentially if $\pi/3 < |\arg z| < \pi$, and it oscillates on the negative real axis (Fig. A). The Airy function has an infinite number of zeroes, all on the real axis.

The functions $Ai(ze^{\pm 2\pi i/3})$ also satisfy the differential equation

$$\frac{d^2y}{dz^2} = zy.$$

Since a second-order differential equation has only two linearly independent solutions, the solutions $Ai(z)$, $Ai(e^{2\pi i/3}z)$, $Ai(e^{-2\pi i/3}z)$, must have a linear relation among them. The power series shows that the relation is

$$Ai(z) = e^{-\pi i/3}Ai(e^{2\pi i/3}z) + e^{\pi i/3}Ai(e^{-2\pi i/3}z).$$

If we start from the equation,

$$\frac{d^2y}{dz^2} - zy = 0$$

and let

$$y = \sqrt{z}\,w(z)$$

we have

$$\sqrt{z}\,\frac{d^2w}{dz^2} + \frac{1}{\sqrt{z}}\,\frac{dw}{dz} - w\left(z^{3/2} + \frac{1}{4z^{3/2}}\right) = 0.$$

Changing the independent variable to $\zeta = (2z^{3/2})/3$, we have

$$\frac{d^2w}{d\zeta^2} + \frac{1}{\zeta}\,\frac{dw}{d\zeta} - w\left(1 + \frac{1}{9\zeta^2}\right) = 0.$$

So w is a modified Bessel function of order 1/3. $w(\zeta) = K_{1/3}(\zeta)/\pi\sqrt{3}$ is the function with the correct asymptotic behavior, so

$$Ai(z) = \frac{1}{\pi\sqrt{3}}\sqrt{z}\,K_{1/3}\left(\frac{2}{3}\,z^{3/2}\right).$$

1. G.B. Airy, "On the Intensity of Light in the Neighborhood of a Caustic," *Trans. Camb. Phil. Soc.* **6**, 379 (1838).

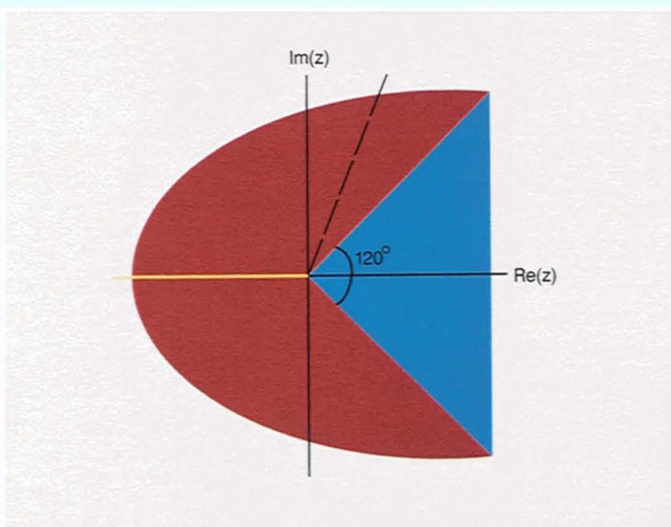

Fig. A — When evaluated in the complex plane, the Airy function gives the regions of exponential decay (blue), exponential growth (red), and oscillation (yellow).

Sir George Biddell Airy (1801-1892)

Airy, the son of a farmer, was a snobbish, self-seeking youngster, who preferred his well-to-do uncle to his father. At the age of 12 he persuaded that uncle to carry him off and bring him up. He entered Cambridge in 1819 and graduated in 1823 at the head of his class in mathematics. He went on to teach both mathematics and astronomy at the university.

He advanced himself with ruthless intensity and, although everyone disliked him, he was successful in his aims. In 1835 he was appointed seventh astronomer royal, a post he was to hold for over 45 years.

He modernized the Greenwich Observatory, equipping it with excellent instruments and bringing it up to the level of the German observatories, which had been forging ahead of those in Great Britain. Airy organized data that had been put aside to gather dust. He was a conceited, envious, small-minded man and ran the observatory like a petty tyrant, but he made it hum.

A strange fatality haunted Airy, causing him to be remembered for his failures. He played the role of the villain of the piece in the failure of J.C. Adams to carry through the discovery of Neptune. In fact, Airy is far better known as the man who muffed the discovery of that planet than for any of the actual accomplishments for which he was deservedly knighted in 1872.

Airy, in 1827, was the first to attempt to correct astigmatism in the human eye (his own) by use of a cylindrical eyeglass lens. He contributed to the study of interference fringes and to the mathematical theory of rainbows. The Airy disk, the central spot of light in the diffraction pattern of a point light source, is named for him.

[Excerpted from I. Asimov, *Asimov's Biographical Encyclopedia of Science and Technology*, Doubleday, Garden City, NY, 1982 and from *Encyclopaedia Britannica*, Fifteenth Edition, 1986.]

$$\int_0^\infty \frac{x^{-1/2} e^{-x} K_\nu(x)}{x+\zeta}\,dx = \frac{\pi e^{\zeta} K_\nu(\zeta)}{\zeta^{1/2}\,[\cos(\nu\pi)]} \tag{3}$$

for $|\arg\zeta| < \pi$ and $\mathrm{Re}(\nu) < \frac{1}{2}$.

Using

$$K_{1/3}(x) = \frac{\pi\sqrt{3}}{\left(\frac{3}{2}x\right)^{1/3}} Ai\left[\left(\frac{3x}{2}\right)^{2/3}\right],$$

we can solve Eq. 3 for $Ai(z)$,

$$Ai(z) = \frac{1}{2}\pi^{-1/2} z^{-1/4} e^{-2/3z^{3/2}} \int_0^\infty \frac{\rho(x)}{1+\frac{3x}{2z^{3/2}}}\,dx$$

for $|\arg z| < 2\pi/3$ and $|z| > 0$, (4)

where $\rho(x)$ is a non-negative exponentially decreasing function,

$$\rho(x) = \pi^{-1/2}\, 2^{-11/6}\, 3^{-2/3}\, x^{-2/3}\, e^{-x} Ai\left[\left(\frac{3x}{2}\right)^{2/3}\right].$$

Substituting $\zeta = (2/3)\, z^{3/2}$ into Eq. 4 we find that

$$Ai(z) = \frac{1}{2}\pi^{-1/2} z^{-1/4} e^{-\zeta} \int_0^\infty \frac{\rho(x)}{1+x/\zeta}\,dx$$

for $|z| > 0$ and $|\arg\zeta| < \pi$.

The moments, μ_k, of $\rho(x)$, needed for the Gaussian quadrature, can be evaluated in closed form

$$\mu_k = \int_0^\infty x^k \rho(x)\,dx = \frac{\Gamma(3k+1/2)}{54^k k!\,\Gamma(k+1/2)} \tag{5}$$

for $k = 0, 1, 2, \ldots.$

The weights, w_i, and abscissas, x_i, can be found for the N-point Gaussian quadrature. The Airy function, when evaluated by the Gaussian quadrature method, depends on N

$$Ai(z, N) \equiv \frac{1}{2}\pi^{-1/2} z^{-1/4} e^{-\zeta} \sum_{i=0}^{N} \frac{w_i(N)}{1+x_i(N)/\zeta}$$

where $\zeta = \frac{2}{3} z^{3/2}$ and $|z| > 0$, $|\arg\zeta| < \pi$. (6)

Setting $N = 10$ was found to provide adequate accuracy. Table 1 gives a list of weights w_i and abscissas x_i for a 10-term Gaussian quadrature integration of the Airy function.

Thus for small z, we use the power series; for large z ($|\arg(z)| \geq 2\pi/3$), we use the quadrature method. A connection formula [Ref. 5, Eq. 10.4.7] transforms a point from the sector $|\arg(z)| \geq 2\pi/3$, where the integral representation is not valid, to a weighted sum of two linearly independent points outside this sector

$$Ai(z) = e^{\pi i/3}\, Ai(ze^{-2\pi i/3}) + e^{-\pi i/3}\, Ai(ze^{2\pi i/3}).$$

The dividing line between the region that should be solved with the power series and the region that should be solved by the quadrature method was determined empirically. Figure 2 shows where the regions on the complex plane lie that are solved, respectively, by the power series expansion, the Gaussian quadrature method, and the connection formula.

Use of Airy Function in Computing Spherical Earth Diffraction

The spherical earth diffraction can be computed by substituting the values of the Airy function computed by this procedure into Eq. 1. There may be floating-point overflows for large values of z, because the Airy function increases exponentially with z, even though the large exponent is offset by the $\exp(\sqrt{3}\, a_n x/2)$ factor of Eq. 1 in the region of interest. Therefore, a scaled Airy function is useful

$$\overline{Ai}(w) = Ai(w) \exp\left(\frac{2}{3} w^{3/2}\right).$$

Equation 1 can therefore be rewritten as

$$F(x,y,z) = 2\sqrt{\pi x} \sum_{n=1}^{\infty} \frac{\overline{Ai}(a_n + ye^{\pi i/3})}{e^{\pi i/3}\, Ai'(a_n)} \; \frac{\overline{Ai}(a_n + ze^{\pi i/3})}{e^{\pi i/3}\, Ai'(a_n)} \cdot$$

$$\exp\left[\frac{1}{2}(\sqrt{3} + i)\, a_n x - \frac{2}{3}(a_n + ze^{\pi i/3})^{3/2} - \frac{2}{3}(a_n + ye^{\pi i/3})^{3/2}\right]. \qquad (7)$$

Equation 7 is evaluated by computing additional terms until two successive terms contribute less than 0.0005. Then, to verify the computation, one more check is performed. If any one of the terms of the series of Eq. 1 contributes more than 10,000 per term, we do not return a value for the propagation factor using Fock's series, since the accuracy of the value found for the Airy function is at least one in 10,000. The maximum contribution of 10,000 per term was chosen because, when the Airy function is evaluated in Eqs. 2, 5, or 6, the accuracy provided is at least four significant digits.

Table 1 — Abscissas and Weights for the 10-Term Gaussian Quadrature Integration for the Airy Function

i	*Abscissas* x_i	*Weights* w_i
1	$8.05943\ 59215\ 34400 \times 10^{-3}$	$8.12311\ 41342\ 35980 \times 10^{-1}$
2	$2.01003\ 46009\ 05718 \times 10^{-1}$	$1.43997\ 92416\ 04145 \times 10^{-1}$
3	$6.41888\ 58403\ 66331 \times 10^{-1}$	$3.64404\ 15851\ 09798 \times 10^{-2}$
4	$1.34479\ 70831\ 39945 \times 10^{0}$	$6.48895\ 66012\ 64211 \times 10^{-3}$
5	$2.33106\ 52313\ 84954 \times 10^{0}$	$7.15550\ 10754\ 31907 \times 10^{-4}$
6	$3.63401\ 35043\ 78772 \times 10^{0}$	$4.43509\ 66599\ 59217 \times 10^{-5}$
7	$5.30709\ 43079\ 15284 \times 10^{0}$	$1.37123\ 91489\ 76848 \times 10^{-6}$
8	$7.44160\ 18468\ 33691 \times 10^{0}$	$1.75840\ 56386\ 19854 \times 10^{-8}$
9	$1.02148\ 85480\ 60315 \times 10^{1}$	$6.63676\ 86881\ 75870 \times 10^{-11}$
10	$1.40830\ 81071\ 97337 \times 10^{1}$	$2.67708\ 43712\ 47434 \times 10^{-14}$

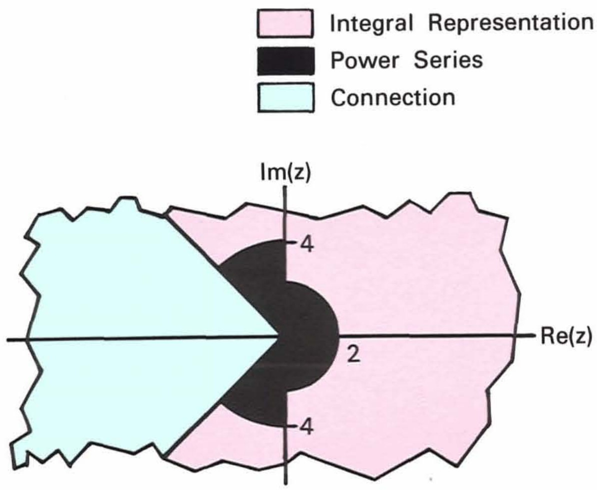

Fig. 2 — The Airy function can be evaluated with three methods: a power series expansion, the Gaussian quadrature method, and a connection formula. The correct choice of evaluation method depends on the region of the complex plane under consideration.

Test of the Model

We tested our calculation by comparing its results with the results of a multipath calculation over a smooth conducting sphere. The results are shown in Fig. 3 for a high-altitude case — the most challenging situation. The dotted curve shows the results of the multipath calculation; the solid curve gives the results of the diffraction calculation.

It can be seen that there is a region (between about 430 km and 450 km) where the two calculations are very close. We can conclude that, for this case, the diffraction calculation was accurate through the intermediate region into the multipath region, for these antenna heights. The diffraction calculation is accurate even within the second multipath lobe in Fig. 3.

We then examined the range of x, y, and z over which our calculation gives accurate results. SEKE uses the diffraction calculation if the minimum clearance of a ray is less than

$$\sqrt{\lambda d_1 d_2/(d_1+d_2)}$$

where λ is the wavelength and $d_1 (d_2)$ is the ground range from the antenna (target) to the point of minimum clearance. We have checked the new diffraction algorithm for all values of y and z less than 5000 and found that it provides sufficient accuracy for all x such that SEKE would use the diffraction calculation. These values of y and z correspond to heights of 375 km at 170 MHz (VHF) or 17 km at 17 GHz (Ku band).

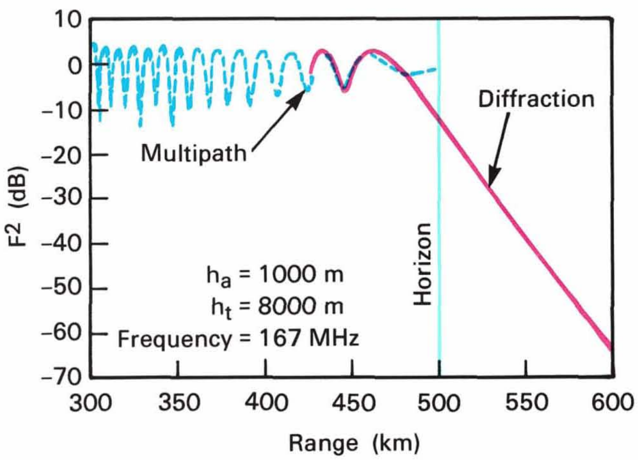

Fig. 3 — The blue curve shows the values of the one-way propagation factor, F^2, obtained from the multipath model. The red curve gives the values of F^2 found by the spherical earth diffraction model.

Summary

The diffraction algorithm increases the accuracy of the calculated propagation factor in the spherical earth diffraction region for a large range of normalized heights. This algorithm requires the application of Fock's series and requires the accurate evaluation of the complex Airy function.

The complex Airy function was evaluated for $|z|$ close to the origin by using a power series expansion. For large $|z|$, the 10-term Gaussian quadrature approximation to an integral representation was used. The calculated Airy function agrees with published tables of the Airy function to within at least four significant digits. Incorporating this program in the SEKE propagation model gives accurate values for the propagation factor to heights of 375 km at VHF or 17 km at Ku band.

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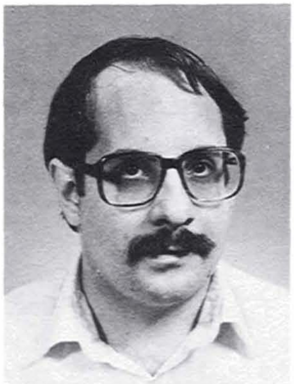

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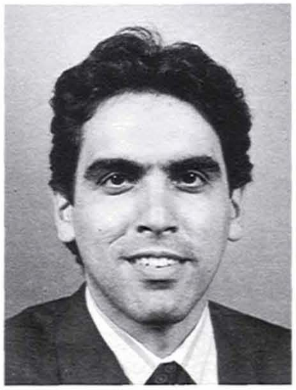

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