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Surveillance Aspects of the Advanced Air Traffic Management System

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16. Abstract

Three topics with impact on the performance of Air-to-Satellite-to-Ground Systems for Air Traffic Control Surveillance are addressed in detail: 1) vulnerability to intentional jamming, 2) performance degradation due to the multiple access noise which results from uncoordinated aircraft transmissions, and 3) tracking techniques for improved surveillance accuracy and reduced short-term outages.

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SECTION I

INTRODUCTION AND SUMMARY OF CONCLUSIONS

This report represents a summary of our analysis of certain key features affecting the performance of Air-to-Satellite-to-Ground Systems for Air Traffic Control (ATC) Surveillance.

For our purposes, these systems are assumed to determine position by multilateration employing at least four satellites. Each aircraft is assumed to periodically transmit a signature having a fixed duration. No coordination between aircraft is assumed.

The work reported on in this report has been motivated by an analysis[1] of two particular candidate systems concepts. Reference [1] highlighted certain deficiencies in these candidate systems. The analysis of certain of these critical issues is the primary purpose of this current report.

Three particular problems are addressed:

(1) Performance vulnerability to intentional interference.

(2) The effect of waveform modulation on the performance loss due to multiple access noise.

(3) Tracking techniques to achieve improved position estimates.

The results of these studies are summarized in this section; detailed analysis will be found in the remainder of this report.

1.1 VULNERABILITY TO INTENTIONAL INTERFERENCE

The two Candidate Air-to-Satellite-to-Ground Surveillance Systems analyzed in Reference [1] were shown to be extremely susceptible to intentional interference, that is, they could be disabled by a jammer which would require "less prime power than a toaster, be easily transportable in a car or small boat, and be within the reach of many hostile political groups..."

The work reported on in Section 2 is directed towards obtaining a general assessment of the susceptibility of Air-to-Satellite-to-Ground Surveillance Systems to intentional interference. To accomplish this objective an upper bound on the required jammer power has been obtained by assuming that the jammer transmits white gaussian noise rather than a more nearly optimized waveform which would in general require less power. The ground processor is assumed to use matched filters to detect presence or absence of each signature through each satellite and is required to determine position by processing the detected output corresponding to the transmission of K successive signature.

In Table 1.1 we list the maximum jammer power required to insure that at least 10% of the number of airborne aircraft are falsely located with a processor designed to achieve a detection probability of at least 0.88. For K less than 100 the required jammer is modest and could probably be assembled at a cost of a few thousand dollars. As K approaches 1000, the cost and complexity of this unsophisticated jammer is such as to prohibit its use by an in-country hostile group. However, since we have only evaluated upper bounds on the maximum required jammer power, we can not determine whether more sophisticated jammers are impractical for such groups.

Table 1.1. Maximum Required Jammer Power Per Satellite for a 10 mJoule Signal Pulse Energy and a 10 MHz Bandwidth.

Transmission Length (K in Pulses)	ERP (dbw)	rf Power (dbw)	Antenna Diameter (feet)	Beamwidth (degrees)
1	33	15	2	15
10	43	20	4	12
100	53	23	8	5
1000	77	40	18	2

1.2 MULTIPLE ACCESS NOISE

It was noted in Reference [1] that only marginal improvements in the performance of the two candidate systems could be realized through increasing the power transmitted from all aircraft. This effect is a consequence of the fact that a major source of interference is the uncoordinated transmission of all aircraft signatures within a constrained bandwidth. The effect of this multiple access noise is similar to that experienced in a radar system due to clutter.

The major effort of Section 3 is directed towards quantitatively assessing the degradation due to multiple access noise and designing waveforms for amelioriating its effect. Toward this end we assumed matched filter detection and restricted our attention to a limited class of transmitted signatures. The major result of Section 3 is the demonstration that even for the best waveforms within this class, the detection performance in the presence of multiple access noise is inferior to the performance which would be predicted assuming only an additive white gaussian noise interference with inband power equal to the multiple access noise power.

This problem area is a complex one with several remaining identifiable potentially high payoff objectives for future research and development. Notable among these are efforts directed toward evolving detectors better than the matched filter, e.g. decision directed processors which attempt to estimate the multiple access noise and subtract it out; efforts directed towards evolving coding and decoding techniques to amelioriate the performance degradation due to poor detector characteristics; and efforts to generalize the results of Section 3 to a less restricted set of signatures.

1.3 TRACKING FOR IMPROVED PERFORMANCE

One of the factors contributing to the projection of poor performance for the two candidate systems analyzed in Reference [1] was the desire to maintain surveillance data during typical maneuvers, coupled with the resulting variation in received signal level at a particular satellites during such maneuvers. These considerations impacted on the system in two dominant ways: (1) satellite constellations were selected to restrict variations in signal level at some expense in increased geometric dillution; and (2) the residue variation in received signal level degrades performance.

Since each aircraft is assumed to carry a reasonably stable oscillator, it is possible to obtain improved position estimates through tracking. In Section 4, a statistical model of the oscillator instability is used as a basis for evolving and evaluating a tracking algorithm. Tracking is examined as a technique for obtaining both (1) improved position estimates; and (2) position estimates during short outages of all but three of the visible satellites. This later effect may occur during maneuver, as a result of adverse satellite look angles.

The examples treated in Section 4, although they ignore flight dynamic constraints, provide some insight into the value of tracking. Specifically, they illustrate that significantly improved position estimates require an airborne standard more stable than the resulting error in measuring time of arrival in the absence of tracking. For example, if the rms error in the time of arrival estimate is 50 nsec, then a significant improvement in the position estimate through time tracking requires a short-term oscillator stability better than 5 parts in 10^8 (assuming a position update rate of one per second). During outages of all but three of the satellites, tracking without excessive error over several 10's of seconds requires an oscillator an order of magnitude more stable.

At this juncture it is impossible to reach any definitive conclusions about the practicality of using tracking to obtain improved position estimation. In particular, although laboratory oscillators are available with a short-term stability of a few parts in 10^9 at a cost of under a few hundred dollars, it does not follow that tracking is practical for low-cost general aviation terminals. It must be noted that the avionics equipment may be subjected to a variety of environmental conditions which are known to have an adverse effect on the stability of crystal oscillators. Considering the questionable maintenance practices for general aviation avionics equipment, we conclude that considerably more effort is required to properly assess the practicality of this technique. This future activity must include a more comprehensive evaluation of the required stability, an assessment of the impact on required ground processing equipment, and an assessment of the practically achievable oscillator stability.

SECTION 2

PERFORMANCE VULNERABILITY TO INTENTIONAL INTERFERENCE

2.1 INTRODUCTION AND SUMMARY OF CONCLUSIONS

In this section, the susceptibility of Air-to-Satellite-to Ground surveillance systems is addressed. For our purposes each aircraft is assumed to periodically transmit a unique signature to four satellites. The satellites relay the signatures to a ground station where they are initially processed using matched filter detectors.

The data obtained in this manner is used first to decide whether or not a given aircraft is in the airspace at a specific time. If this decision is affirmative, the data is then used to decide what the aircraft's position was at the relevant time. This two stage decision process is carried out for each aircraft at the end of every signature repetition period. It is assumed that the ground station employs some tracking algorithm in order to carry out this decision process. Specifically, in making each decision for a given aircraft, the ground station utilizes the data supplied to it during the preceding K signature repetition periods by the matched filter detectors.

We assume that this surveillance system is operating in the presence of an intentional jammer transmitting in band white gaussian noise. In general, an intelligent jammer could by appropriately selecting the modulation be more effective with less power.

In order to study the variation of system performance with both system and jammer variables, a measure of system performance must be defined. The parameters $\overline{N}_F$, $\overline{N}_{AD}$ and P_D were used to measure system performance. $\overline{N}_F$ is the expected number of false alarms generated by the ground station during a single signature repetition period. $\overline{N}_{AD}$ is the expected number of ambiguous detections generated by the ground station during a single signature repetition period. P_D is the probability that a given aircraft is detected during a signature repetition period.

A false alarm occurs on a signature repetition period if the ground station declares that a given aircraft is in the airspace (at the time relevant to the decision period), but in fact the aircraft is not in the airspace. A given aircraft is declared detected on a signature repetition period if the ground station declares that it is in the airspace and in fact it is. An aircraft is declared ambiguously detected on a signature repetition period if it is detected, but has an incorrect position decided for it on the second stage of the decision process.

Performance was actually analyzed by determining $\overline{N}_{FL}$ and $\overline{N}_{ADL}$, lower bounds to $\overline{N}_F$ and $\overline{N}_{AD}$, for a fixed value of P_{DL}. P_{DL} is a lower bound to P_D. Parametric expressions for these lower bounds were derived and were then evaluated for a representative set of parameter values, i.e., 10% of the total aircraft population airborne, a one second signature repetition rate and a 10 MHz bandwidth at L band.

The results of our analysis indicated that performance parameters exhibit a strong threshold behavior as a function of signal to interference ratio; for illustrative purposes we have selected the threshold values as(0.9) N_T for $\overline{N}_F$ and (0.01) N_T for $\overline{N}_{AD}$ where N_T is the total aircraft population.

A brief summary of the results are presented in Table 1.1.

2.2 OUTLINE OF THE ANALYSIS

As has already been noted our goal in this report is to study the susceptibility of Air-to-Satellite-to-Ground surveillance systems to jamming. The program of this effort will be as follows. In Section 2.3 the class of Air-to-Satellite-to-Ground surveillance systems will be described in detail. In Section 2.4 the parameters used to measure the performance of this representative system will be defined. In Section 2.5 preliminary performance bounds will be derived. In Section 2.6 we shall describe the intentional interference. The remaining sections will be concerned with developing the desired performance bounds. The variation of these bounds with both system and jammer parameters will be studied.

2.3 DESCRIPTION OF THE SYSTEM

In this section we shall define the class of Air-to-Satellite to-Ground systems on which we shall concentrate our efforts. For our purpose, each aircraft is assumed to transmit a unique signature periodically with period α. These signatures are assumed to be received by four satellites and transmitted noiselessly to a ground station where matched filters are employed to detect the signatures and to estimate arrival time differences between satellites and hence aircraft position. The output of each matched filter is to be sampled at twice the Nyquist rate.

In actual practice the phase of the transmitted signatures will not be known at the ground station and thus "matched filter-envelope detectors" would be used instead of "matched filter detectors." However, in light of this, our assumption of matched filter detectors is not inconsistent with our analysis for the following reason. The performance of a matched filter detector is always better than a "matched filter-envelope detector" since phase is assumed known. Our aim of course is the determination of upper bounds to required jammer power. Any such bounds determined under a "matched filter detector assumption" will of course be valid under a "matched filter-envelope detector assumption." We shall also neglect up-link noise and multiple access noise since our stated goal is upperbounding the required jamming power.

The ground processor is assumed to partition the sequence of output samples from each of the 4 N_T matched filters into time segments of length $\alpha + \bar{\beta}$ where N_T is the total number of aircraft and $\bar{\beta}$ is the maximum delay between the first and last reception of a particular signature at the constellation of four satellites. If an aircraft is present in the airspace, its transmitted signature will be received at all four filters matched to it, at the ground, at least once in each such time segment and possibly twice, no matter what origin is chosen for the initial segment. This is desirable, since it implies that the ground station can begin processing whenever it wants to. The possible presence of a signature more than once in a segment will cause no harm, as will be evident.

Consider some time origin set up at the ground station and let Seg. (n) be the nth time segment of length $\alpha + \overline{\beta}$ seconds set up relative to the time origin. A ground processor starts with Seg. (1) and observes the outputs of the four filters (one for each satellite) which are matched to aircraft "i's" signature. For each of the filters, it lists those time points in Seg. (1) at which the matched filter-detector declared the output sample to have been generated by reception of aircraft "i's" signature. The ground processor takes this data and computes the aircraft positions implied by all possible time differences generated by the listed time points. The ground processor is assumed to eliminate any unrealistic positions, i.e., time differences. The processor lists the implied positions on a list we shall call "List (i,1)." One should note that this list may be blank.

The processor repeats the procedure just described for the next (K-1) time segments; Seg. (2)...Seg. (K). At the end of these K segments the processor enters a two stage decision procedure. On the first stage of the procedure it decides the following question. Is aircraft "i" in the airspace at the end of the Kth time segment (or at some other relevant time interior to the K time segments)?

We define now the following hypotheses:

$H_o(i)$ = aircraft "i" is not in the airspace at the end of the Kth time segment (or at the relevant interior time)

$H_1(i)$ = aircraft "i" is in the airspace at the end of the Kth time segment (or at the relevant interior time)

If $H_o(i)$ is true, yet the processor decides that $H_1(i)$ is true, we say that an aircraft "i" false alarm has occurred on the Kth time segment (or at the relevant interior time). If hypothesis $H_1(i)$ is true and the ground processor decides $H_1(i)$ we say that aircraft "i" has been detected on the Kth time segment (or at the relevant interior time). If hypothesis $H_1(i)$ is true and the ground processor decided $H_o(i)$, we say that aircraft "i" has been missed on the Kth time segment.

If the first stage of the decision procedure resulted with a decision that aircraft "i" was in the airspace then the second stage of the decision procedure is entered. On this second stage the ground processor decides the following question: What is the position of aircraft "i" during the Kth segment (or at the relevant interior time)? In order to be able to make this decision the ground processor is supplied with a set of possible position candidates, called "the Position Set of aircraft 'i' on the Kth segment." This position set is generated in some manner from {List (i,1)...List (i,K)}. Hopefully, (if aircraft "i" is in the airspace) there will be only one position in the position set, the correct position. If there is more than one position, the ground processor chooses one at random and supplies it to a central surveillance station.

Consider the following definitions

$Pos_K(i)$ = true position of aircraft "i" (given $H_1(i)$ is true) during Seg. (K) (or the relevant time)

$\{p_1(i),...p_{jk}(i)\}$ = position set of aircraft "i" during Seg. (K)

$p_k^*(i)$ = aircraft "i" position decided upon during the second stage of the decision process at the end of Seg. (K)

If $H_1(i)$ is true and decided and $p_k^*(i)$ equals $Pos_k(i)$, we say that aircraft "i" has been <u>completely detected</u> on Seg. (K). If $H_1(i)$ is true and decided and $p_k^*(i)$ is not equal to $Pos_k(i)$, we say that aircraft "i" is <u>ambiguously detected</u>.

The actual mechanics of the first and second stage decision procedures will be kept with minimum specification in order to keep the system as general as possible. The first stage procedure will be discussed with somewhat more exactness later in Appendix A. The second stage decision procedure will not be specified any further. The two stage procedure essentially uses some tracking algorithm on K segments worth of data in order to detect aircraft and determine their position. Although we will keep these decision procedures as general as possible, we insist that they always act rationally. To be precise they obey the following axiom:

<u>Rationality Axiom</u>

If there exists a sequence of K positions; one on List (i,1), one on List (i,2),...one on List (i,K) and if this sequence of positions looks as if it might be that of an aircraft in flight (i.e. the successive positions obey the constraints of flight dynamics), then on the first stage of the decision procedure it must be decided that aircraft "i" is in the airspace, and a position generated by this sequence of K positions must appear in the position set of aircraft "i".

Before, it was noted that a received signature may occur twice in the same time segment. If this occurs on all four filter outputs, it will merely cause two received signature quadruplets to be mapped into the same position on "List (i,)." If this does not occur on all four filter outputs at worst it will generate erroneous positions on "List (i,)". In neither case will it cause any processing ambiguity for the surveillance system.

As has already been stated, an aircraft position is computed using three time differences. Without loss of generality we view the lists {List (i,1),...List (i,K)} as storing triplets of time differences rather than positions. The second stage decision would of course then be on a triplet of time differences. We will assume from now on that the representative surveillance system makes its decisions on the time difference triplets instead of on positions.

After the ground processor completes the decision process at the end of Seg. (K) it repeats this procedure using; Seg. (2),...Seg. (K + 1), then Seg.(3),...Seg. (K+2) etc. Because of time invariance we can judge the system just by its operation on Seg. (1),... Seg.(K).

2.4 PERFORMANCE PARAMETERS

We define the aircraft "i" detection probability, $P_D(i)$, to be the probability of deciding $H_1(i)$ given $H_1(i)$ is true. Since all links are assumed to be identical

$$P_D \doteq P_D(i) \text{ for all } i$$

We shall measure the performance by determining lower bounds to the average number of false alarms, $\overline{N}_F$ and the average number of ambiguous detections $\overline{N}_{AD}$ with P_D held fixed (or equivalently kept above) some value. The effect of jamming will be analyzed by studying the variation of these lower bounds with both system and jammer parameters.

The following two parameters; p_f and p_d, are measures of the performance of the matched filter detector.

$$p_f = \text{Prob}\left(\begin{array}{l}\text{the matched filter}\\ \text{detector declares the sample}\\ \text{to have been generated by}\\ \text{a received aircraft signa-}\\ \text{ture and interference}\end{array}\middle|\begin{array}{l}\text{sample was only}\\ \text{generated by}\\ \text{interference}\end{array}\right)$$

$$p_d = \text{Prob}\left(\begin{array}{l}\text{the matched filter}\\ \text{detector declares the sample}\\ \text{to have been generated by}\\ \text{a received aircraft signa-}\\ \text{ture and interference}\end{array}\middle|\begin{array}{l}\text{sample was generated}\\ \text{by a received aircraft}\\ \text{signature and inter-}\\ \text{ference}\end{array}\right)$$

We describe the first stage decision process operating by deciding aircraft "i" in the airspace if more than Kt of the Lists; List (i,1),...List (i,K), have at least one entry. The dependence of P_D on p_d through a lower bound to it has been derived in Appendix A to be of the form

$$P_D \geq p_d^4 \quad \text{when } K = 1 \qquad (2.4.1)$$

$$P_D \geq 1 - \exp -K(\nu(1-t) - Ln((1-p_d^4)e^{\nu} + p_d^4)) \text{ when } K > 1 \qquad (2.4.2)$$

Where t is ε [0,1] and ν is the nonnegative number which maximizes

$$\nu (1 - t) - Ln ((1-p_d^4)e^{\nu} + p_d^4)$$

The parameters "t" and "p_d" may be varied jointly to allow the right hand side of (2.4.2) to attain some value. We shall refer to the right hand side of (2.4.1)-(2.4.2) as P_{DL} in which case we have

$$P_D \geq P_{DL}$$

For a given desired value of P_D our surveillance system matched filter detector and first stage decision process will be designed with a "t" and "p_d" such that P_{DL} equals this desired value, thus guarranteeing that P_D will attain it. Of course when K = 1, t does not enter the design.

2.5 PRELIMINARY LOWER BOUNDS TO $\overline{N}_F$ AND $\overline{N}_{AD}$

In this section we shall derive some preliminary lower bounds to $\overline{N}_F$ and $\overline{N}_{AD}$. These bounds will be expanded in Section 2.6.

2.5.1 Preliminary Lower Bound to $\overline{N}_F$

We begin by defining the following indicator functions

$\theta_F(i)$ = 1, if aircraft "i" is not in the airspace on the Kth segment (or at the relevant time, interior to the K time segments)

$\theta_F(i)$ = 0, otherwise

$\gamma_F(i)$ = 1, if $\theta_F(i)=1$ and the first stage decision declares aircraft "i" to be in the airspace of the relevant time

$\gamma_F(i)$ = 0, otherwise

Clearly the number of false alarms is

$$n_F = \sum_{i=1}^{N_T} \theta_F(i)\, \gamma_F(i)$$

and*

$$\overline{N}_F = \underset{\sim}{E}(n_F) = \underset{\sim}{E}\left(\sum_{i=1}^{N_T} \theta_F(i)\, \gamma_F(i)\right)$$

$$\overline{N}_F = N_T\, \underset{\sim}{E}(\theta_F(i)\, \gamma_F(i))$$

$$\overline{N}_F = N_T\, P(\gamma_F(i) = 1 \mid \theta_F(i) = 1)\, P(\theta_F(i) = 1) \qquad (2.5.1)$$

where N_T is the total aircraft population, i.e., number of different signatures Of course,

$$P(\theta_F(i) = 1) = 1 - \frac{N}{N_T} \qquad (2.5.2)$$

---

* $\underset{\sim}{E}$ is the expectation operator

where N is the number of airborne aircraft.

Let us define

$$P_F = P\ (\gamma_F(i) = 1 \mid \theta_F(i) = 1) \qquad (2.5.3)$$

A lower bound to P_F is derived in Appendix B. We now quote it

$$P_F \geq \exp\left[(K\ t + 1)\ \mathrm{Ln}\left(\left(1 - (1-p_f)^{2B\alpha}\right)\left(1-(1-p_f)^{4B\beta}\right)^3\right)\right] \qquad (2.5.4)$$

where β is the delay between the first and last reception of a particular signature at the constellation of four satellites and B is the bandwidth. Applying Eqs. 2.5.2, 3 and 4 to 2.5.1 we obtain

$$\overline{N}_f \geq \overline{N}_{FL} \doteq (N_T - N)\ \exp\left[(K\ t + 1)\mathrm{Ln}\left(\left(1-(1-p_f)^{2B\alpha}\right)\left(1-(1-p_f)^{4B\beta}\right)^3\right)\right] \qquad (2.5.5)$$

which is the desired preliminary lower bound*.

2.5.2 Preliminary Lower Bound to $\overline{N}_{AD}$

As in the previous subsection we begin by defining several indicator functions.

$\theta_D(i)$ = 1, if aircraft "i" is in the airspace on the Kth segment (or the relevant time interior to the K time segments)

$\theta_D(i)$ = 0, otherwise

$\Psi_D(i)$ = 1, if given that $\theta_D(i) = 1$, aircraft "i" is detected on the Kth time segment, Seg. (K)

---

* It is understood that when K = 1 t = 0 (the only logical value for t*).

$\Psi_D(i)$ = 0, otherwise

$\gamma_D(i)$ = 1, if there is a first sequence of K arrival time difference triplets with the first sequence member on List (i,1),... the Kth sequence member on List (i,K). The elements of this sequence have all their component arrival times, T( ), only generated by interference (not interference plus aircraft "i's" signature). This sequence of time difference triplets appears as if it were generated by an aircraft in flight.

$\gamma_D(i)$ = 0, otherwise

$\Gamma_D(i)$ = 1, if given that $\gamma_D(i)=1$, an incorrect position will be picked from the Position Set of aircraft "i" on the Kth segment

$\Gamma_D(i)$ = 0, otherwise

The following inequality on the number of ambiguous detections is evident.

$$n_{AD} \geq \sum_{i=1}^{N_T} \Gamma_D(i)\, \gamma_D(i)\, \Psi_D(i)\, \theta_D(i) \qquad (2.5.6)$$

This is an inequality rather than an equality since an aircraft "i" ambiguous detection is still possible even if $\gamma_D(i)$ equals zero. From Eq. 2.5.6 the following inequality is immediately obtained.

$$\overline{N}_{AD} = \underset{\sim}{E}(n_{AD}) \geq \sum_{i=1}^{N_T} \underset{\sim}{E}(\Gamma_D(i)\, \gamma_D(i)\, \Psi_D(i)\, \theta_D(i))$$

$$\overline{N}_{AD} \geq N_T \text{ Prob } (\Gamma_D(i) = 1,\ \gamma_D(i) = 1,\ \Psi_D(i) = 1,\ \theta_D(i) = 1) \qquad (2.5.7)$$

$$N_{AD} \geq N_T \text{ Prob } (\Gamma_D(i) = 1 \mid \gamma_D(i) = 1,\ \Psi_D(i) = 1,\ \theta_D(i) = 1) \text{Prob } (\gamma_D(i) = 1)$$

$$\text{Prob } (\Psi_D(i)=1 \mid \theta_D(i) = 1)\ P_{rob}(\theta_D(i)=1) \qquad (2.5.8)$$

Each of the terms on the right hand side of Eq. 2.5.8 will now be lower bounded:

Obviously,

$$\text{Prob}\ (\theta_D(i) = 1) = \frac{N}{N_T} \tag{2.5.9}$$

Of course,

$$\text{Prob}\ (\psi_D(i) = 1 |\ \theta_D(i) = 1) = P_D \geq P_{DL}$$

Hence, we obtain immediately from Eqs. 2.4.1 and 2.

$$\text{Prob}\ (\Psi_D(i) = 1\ |\ \theta_D(i) = 1) \geq p_d^{\ 4} \text{ when } K = 1$$

$$\text{Prob}\ (\Psi_D(i) = 1 |\ \theta_D(i) = 1) \geq 1 - \exp -K(\nu(1-t) - \text{Ln}\ ((1 - p_d^{\ 4})\ e^{\nu} + p_d^{\ 4})) \quad \text{when } K > 1$$

In Sec. 2.3, a Rationality Axiom was stated. We insisted that the ground processor, whatever its design must obey this axiom. If the event $\{\gamma_D(i) = 1\}$ occurs then the Rationality Axiom implies that there will be at least one incorrect position in the Position Set of aircraft "i". In Sec. 2.3 we also stated that if there was more than one position in the Position Set of aircraft "i", then the ground processor would pick one position from the set at random and assume it was the correct position of aircraft "i". The event $\{\gamma_D(i) = 1\}$ being true thus implies that with probability greater than 0.5, an incorrect position will be picked and we have

$$\text{Prob}\ (\Gamma_D(i) = 1\ |\ \gamma_D(i) = 1,\ \ \Psi_D(i) = 1,\ \theta_D(i) = 1) \geq 0.5 \tag{2.5.10}$$

We have yet to lower bound Prob $(\gamma_D(i) = 1)$. This is quite a complicated procedure and is carried out in Appendix C. We quote the result.

$$\text{Prob}\ (\gamma_D(i) = 1) \geq \exp\left(\begin{array}{l} (K-1)\ \text{Ln}\left(\left(1-(1-p_f)^{2B\alpha}\right)\left(1-(1-p_f)^{\frac{8B}{3}\frac{v\alpha}{c}}\right)^3\right) \\ + \text{Ln}\left(\left(1-(1-p_f)^{2B\alpha}\right)\left(1-(1-p_f)^{4B\beta}\right)^3\right) \end{array} \right) \tag{2.5.11}$$

where v is the maximum aircraft velocity and c is the velocity of light.

Inequalities in Eqs. 2.5.9-11 can be applied to Eq. 2.5.8 to yield

$$\overline{N}_{AD} \geq 0.5\ N\ p_d^4\ (1-(1-p_f)^{2B\alpha})(1-(1-p_f)^{4B\beta})^3 \quad \text{when } K = 1$$

$$N_{AD} \geq 0.5N\ (1-\exp-K(\nu(1-t)-\text{Ln}((1-p_d^4)e^{\nu} + p_d^4)))$$

$$\times$$

$$\exp\left(\begin{array}{l} (K-1)\ \text{Ln}((1-(1-p_f)^{2B\alpha})(1-(1-p_f)^{\frac{8B}{3}\frac{v\alpha}{c}})^3) \\ + \text{Ln}\left((1-(1-p_f)^{2B\alpha})(1-(1-p_f)^{4B\beta})^3\right) \end{array} \right) \tag{2.5.12}$$

$$\text{when } K > 1$$

which is the desired preliminary lower bound. We shall refer to the right hand side of Eq. 2.5.12 as $\overline{N}_{ADL}$.

We desire to expand the expressions for $\overline{N}_{FL}$, $\overline{N}_{ADL}$ and $\overline{P}_{DL}$ obtained and study their variation with both system and jammer parameters. In order to do this we must describe precisely the interference perturbing the surveillance system performance.

2.6 INTERFERENCE AND SIGNATURE ASSUMPTIONS

There are a variety of sources of unintentional interference which perturb the performance of an Air-to-Satellite-to-Ground Surveillance System. Thermal noise and multiple access noise are two such sources of unintentional interference. In addition to these sources of unintentional interference we assume that a jammer is operating and trying to intentionally cause system degradation. The jammer is transmitting white gaussian noise over the transmission bandwidth, B, at each of the four satellites. The jammer generated noise transmitted at each satellite has an ERP (Effective Radiated Power) of J watts.

Out ultimate goal is to study the variation of $\overline{N}_F$ and $\overline{N}_{AD}$ with jammer parameters. We desire, for a given jamming power, to determine lower bounds to $\overline{N}_F$ and $\overline{N}_{AD}$; $\overline{N}_{FL}$ and $\overline{N}_{ADL}$. We shall ignore all types of interference other than jammer generated noise, in computing $\overline{N}_{FL}$ and $\overline{N}_{ADL}$. These will be no less valid as lower bounds since the presence of other sources of interference can only increase $\overline{N}_{FL}$ and $\overline{N}_{ADL}$. Similarly, we shall ignore any degradation that a transmitted signature might suffer due to: incoherence at the receiver, doppler losses, banking of an aircraft during transmission. The presence of any such degradation can only increase $\overline{N}_{FL}$ and $\overline{N}_{ADL}$.

Let $s_i(t)$ be aircraft "i's" signature. We shall represent it as

$$s_i(t) = \sqrt{E}\, S_i(t)$$

where $S_i(t)$ has time duration τ and unit energy. E is the energy of $s_i(t)$. The average power of $s_i(t)$ during transmission is $P_o = E/\tau$. At the ground station there are four matched filters matched to aircraft "i's" signature. We assume that the impulse response of each of these filters is $S_i(-t+\tau)$.

Since both the jammer generated noise and aircraft transmitted signatures will suffer the same range loss in transmission, we shall ignore this loss in any ensuing analyses.

2.7 ANALYSIS OF $\overline{N}_{FL}$ AND $\overline{N}_{ADL}$ WHEN K = 1

We have determined P_{DL}, a lower bound to P_D, given by Eq. (2.4.1)-(2.4.2). We have also determined $\overline{N}_{FL}$ and $\overline{N}_{ADL}$, lower bounds to $\overline{N}_F$ and $\overline{N}_{AD}$, given by Eqs. 2.5.5 and 12. In this section we shall analyze the variation of these bounds with parameters of interest for the special case when K = 1, which is equivalent to operating without tracking.

As has already been stated we assume that the general surveillance system is designed to operate with P_D exceeding some specified value. This minimum value is guarranteed by fixing p_d and t so that P_{DL} has this minimum value. Assume that this has been done and that p_d is fixed. Relative to this fixed value of p_d we have the following expression for p_f derived in Appendix D.

$$p_f = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{1}{2} \frac{P_o B\tau}{J}} - \phi \right) \qquad (2.7.1)$$

where ϕ is the solution of

$$p_d = \frac{1}{2} + \frac{1}{2} \operatorname{erf}(\phi) \qquad (2.7.2)$$

J is the jammer power and B is the bandwidth.

Applying Eqs. (2.7.1) and 2 to Eqs. 2.5.5 and 12 yields

$$N_{FL} = (N_T - N)\left[\left(1 - \left(\frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{1}{2} \frac{P_o B\tau}{J}} - \phi\right)\right)^{2B\alpha}\right)\left(1 - \left(\frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{1}{2} \frac{P_o B\tau}{J}} - \phi\right)\right)^{4B\beta}\right)^3\right]$$

$$N_{ADL} = 0.5N\, p_d^4\left[\left(1 - \left(\frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{1}{2} \frac{P_o B\tau}{J}} - \phi\right)\right)^{2B\alpha}\right)\left(1 - \left(\frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{1}{2} \frac{P_o B\tau}{J}} - \phi\right)\right)^{4B\beta}\right)^3\right]$$

(2.7.3)

as the lower bounds of interest when K = 1. p_d is picked so that P_{DL} (which is p_d^4 when K = 1) has some desired value. We evaluate these lower bounds assuming 10% of the aircraft population is airborne ($N = 0.1N_T$), a 10 MHz bandwidth (B), a one second pulse repetition period (α), and a differential satellite delay (β) of 24 msec. Figures 2.1 and 2.2 illustrate these computed lower bounds plotted as functions of $\frac{J}{P_o B\tau}$ for various values of P_{DL}.

In observing these figures we see that as $\frac{J}{P_o B\tau}$ is decreased from -12 db these lower bounds maintain their maximum values over quite a large range of $J/P_oB\tau$'s. Ultimately, each curve decreases very rapidly as $\frac{J}{P_o B\tau}$ is lowered past a certain value; hence there is a threshold effect. Certainly, any Air-to-Satellite-to-Ground Surveillance System suffering an $\overline{N}_F$ of 0.9 (N_T) and an $\overline{N}_{AD}$ of 0.01 (N_T) has to be considered inoperative. Since these are close to the threshold, we shall use these values as indicators of total system performance degeneration.

Assuming a transmitted energy per pulse of 10 mJoules (e.g. P_o = 30 dbw, τ = 10 μsec), we can determine, from Figs. 2.1 and 2.2, the maximum required jammer ERP, J, that the jammer would have to transmit at each of the four satellites in order to achieve the threshold levels. Since the difference between required jamming power for a P_{DL} of 0.9 and 0.999 is less than 2 db, we shall thus take as a nominal jamming level 33 dbw ERP. This could be realized with 30 watts of rf power and an antenna with a 20^o beamwidth. Clearly a system, without tracking, (K=1) is simple to disable.

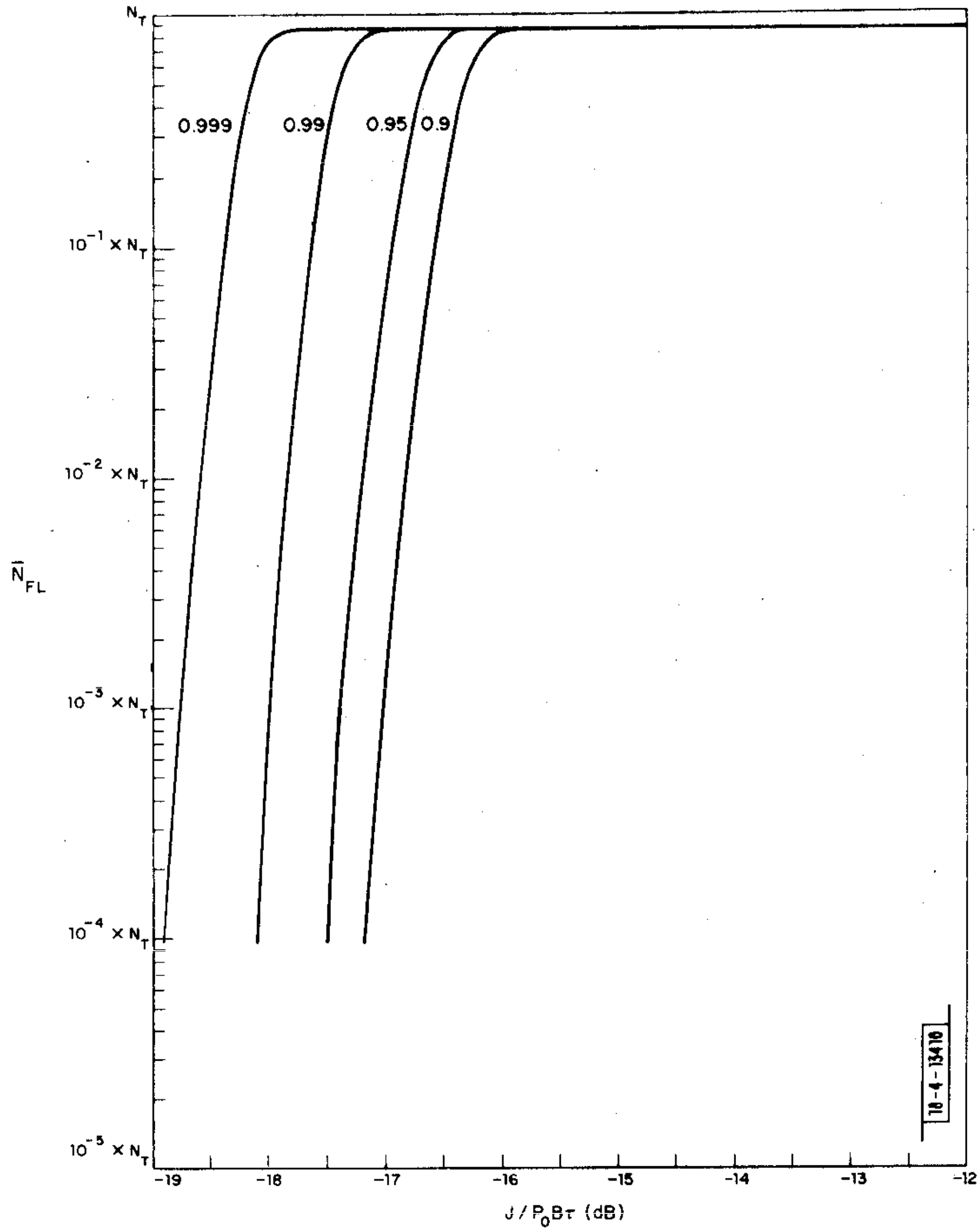

Fig. 2.1. $\overline{N}_{FL}$ vs $\frac{J}{P_oB\tau}$ when K = 1, for various values of P_{DL}.

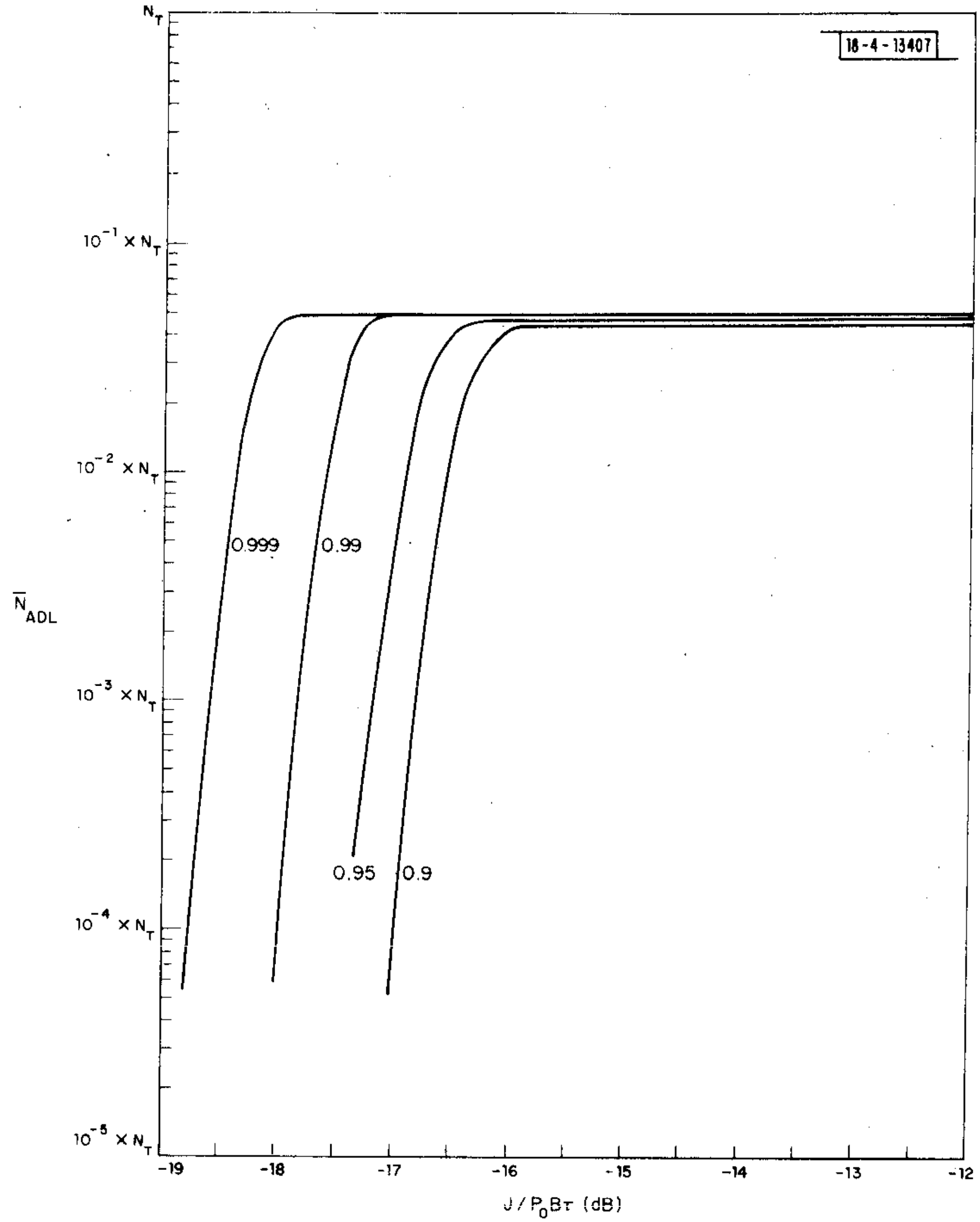

Fig. 2.2. $\overline{N}_{ADL}$ vs $\frac{J}{P_o B\tau}$ when $K=1$, for various values of P_{DL}.

2.8 ANALYSIS OF $\overline{N}_{FL}$ AND $\overline{N}_{ADL}$ WHEN K $\gg$ 1

In this section we shall analyze the variation of the lower bounds, $\overline{N}_{FL}$ and $\overline{N}_{ADL}$, for the case in which the tracking length K, is greater than 1. The preliminary lower bounds for this case are given by Eq. 2.5.5 and 2.5.12. As has already been noted P_{DL} is fixed in the design of the surveillance system. For the case in which K is greater than 1, P_{DL} is given by Eq. 2.4.2 as

$$P_{DL} = 1 - \exp -K\,(\nu(1-t) - Ln((1 - p_d^{\,4})\, e^{\nu} + p_d^{\,4}))$$

A required P_{DL} can be obtained by varying the (t, p_d) pair.

Assume that P_{DL} is obtained by a specific (t, p_d) pair. In Appendix D the following expression is obtained for the p_f suffered by using p_d.

$$p_f = \frac{1}{2} - \frac{1}{2}\ \mathrm{erf}\left(\sqrt{\frac{1}{2}\ \frac{P_0 B\tau}{J}} - \phi\right) \qquad (2.8.1)$$

where ϕ is the solution of

$$p_d = \frac{1}{2} + \frac{1}{2}\ \mathrm{erf}\,(\phi) \qquad (2.8.2)$$

Applying Eqs. 2.8.1 and 2 to Eqs. 2.5.5 and 12 yields

$$\overline{N}_{FL} = (N_T - N)\,\exp\left[(Kt + 1)\,Ln\left(\left(1-\left(\frac{1}{2} + \frac{1}{2}\ \mathrm{erf}\left(\sqrt{\frac{1}{2}\ \frac{P_0 B\tau}{J}} - \phi\right)\right)^{2B\alpha}\right)\left(1 - \left(\frac{1}{2} + \frac{1}{2}\ \mathrm{erf}\left(\sqrt{\frac{1}{2}\ \frac{P_0 B\tau}{J}} - \phi\right)\right)^{4B\beta}\right)^{3}\right)\right] \qquad (2.8.3)$$

$$\overline{N}_{ADL} = (0.5)(N)P_{DL} \exp\left((K-1)\,\mathrm{Ln}\left(\left(1-\left(\frac{1}{2}+\frac{1}{2}\,\mathrm{erf}\left(\sqrt{\frac{1}{2}\frac{P_oB\tau}{J}}-\phi\right)\right)^{2B\alpha}\right)\left(1-\left(\frac{1}{2}+\frac{1}{2}\,\mathrm{erf}\left(\sqrt{\frac{1}{2}\frac{P_oB\tau}{J}}-\phi\right)\right)^{\frac{8B}{3}\frac{v\alpha}{c}}\right)^3\right) + \mathrm{Ln}\left(\left(1-\left(\frac{1}{2}+\frac{1}{2}\,\mathrm{erf}\left(\sqrt{\frac{1}{2}\frac{P_oB\tau}{J}}-\phi\right)\right)^{2B\alpha}\right)\left(1-\left(\frac{1}{2}+\frac{1}{2}\,\mathrm{erf}\left(\sqrt{\frac{1}{2}\frac{P_oB\tau}{J}}-\phi\right)\right)^{4B\beta}\right)^3\right)\right) \qquad (2.8.4)$$

These lower bounds have been computed for various values of K and P_{DL}. This was performed assuming the same system parameter values quoted in Section 2.7 and a maximum velocity, v, of 600 mph. Figures 2.3 through 2.16 illustrate the computed lower bounds plotted as functions of $\frac{J}{P_oB\tau}$ (with K and P_{DL} fixed). We shall spend the remainder of this section discussing these curves.

In observing each set of curves in Figures 2.3 through 2.16 notice that each curve exhibits a strong thresholding property. A given value of P_{DL} may be realized by various (p_d, t) combinations. "p_d" is the matched filter pulse detection probability. "t" is a parameter of the first stage decision process (i.e., an aircraft is decided present if more than Kt lists have at least one entry). In observing each pair of curve sets, $\overline{N}_{FL}$ vs. $\frac{J}{P_oB\tau}$ and $\overline{N}_{ADL}$ vs. $\frac{J}{P_oB\tau}$, one notices that it is more difficult for the jammer to cause system degradation, both in false alarms and ambiguous detections, if a low value of t is used. The reason for this is transparent in the case of ambiguous detections, by the following argument. Suppose

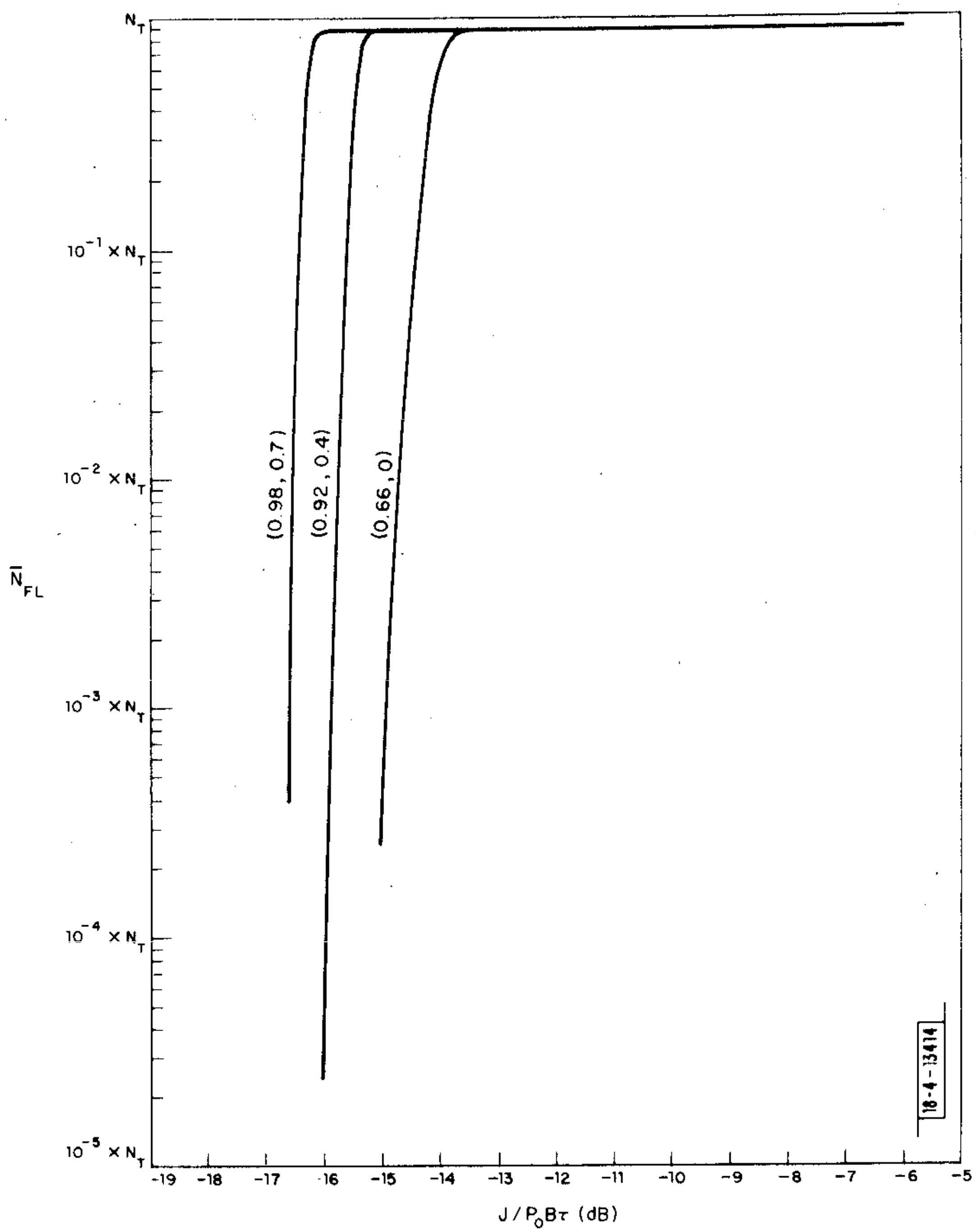

Fig. 2.3. $\overline{N}_{FL}$ vs $\frac{J}{P_o B\tau}$ when $K = 10$, $P_{DL} = 0.88$, for various (p_d, t) combinations.

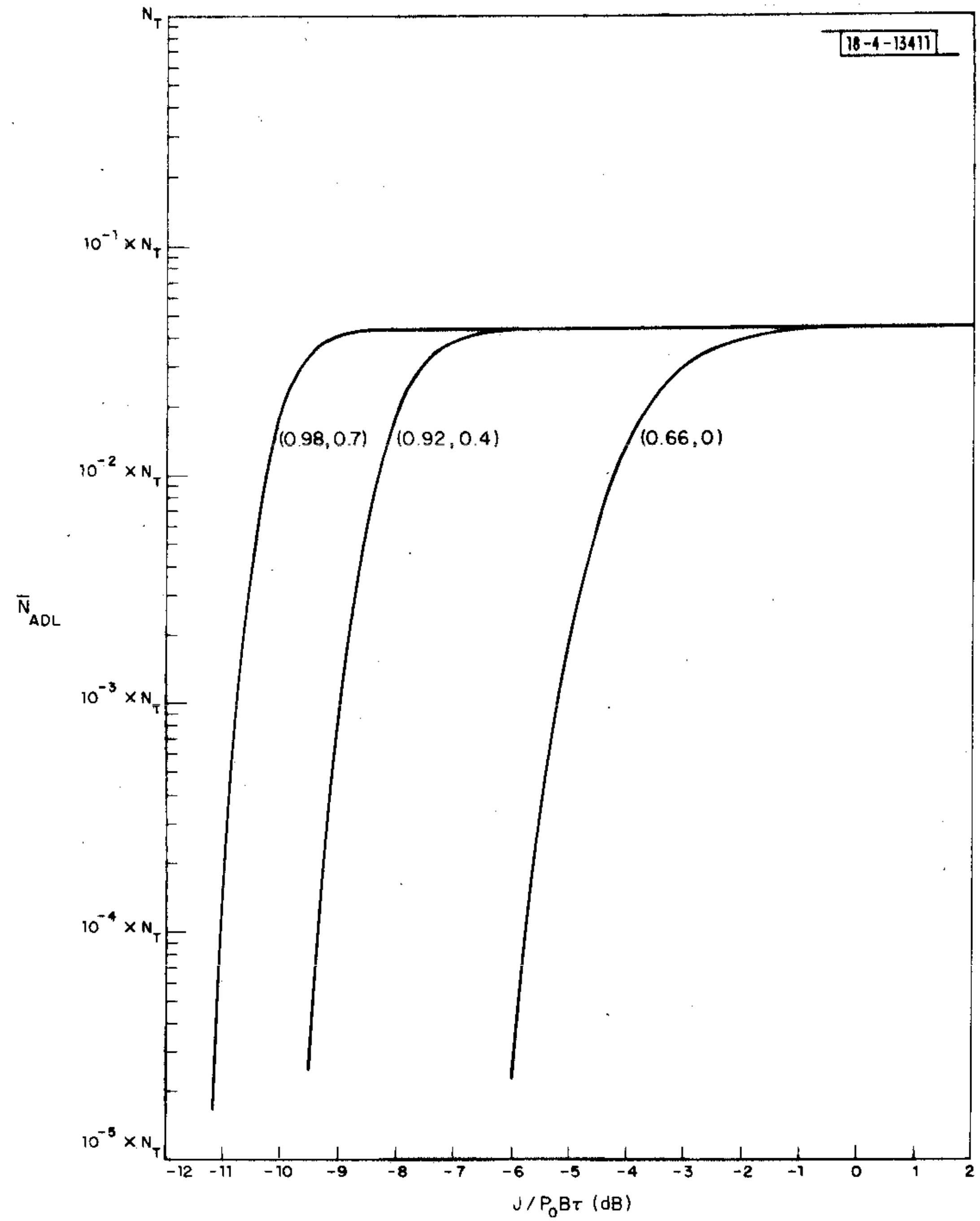

Fig. 2.4. $\overline{N}_{ADL}$ vs $\frac{J}{P_o B\tau}$ when $K = 10$, $P_{DL} = 0.88$, for various (p_d, t) combinations.

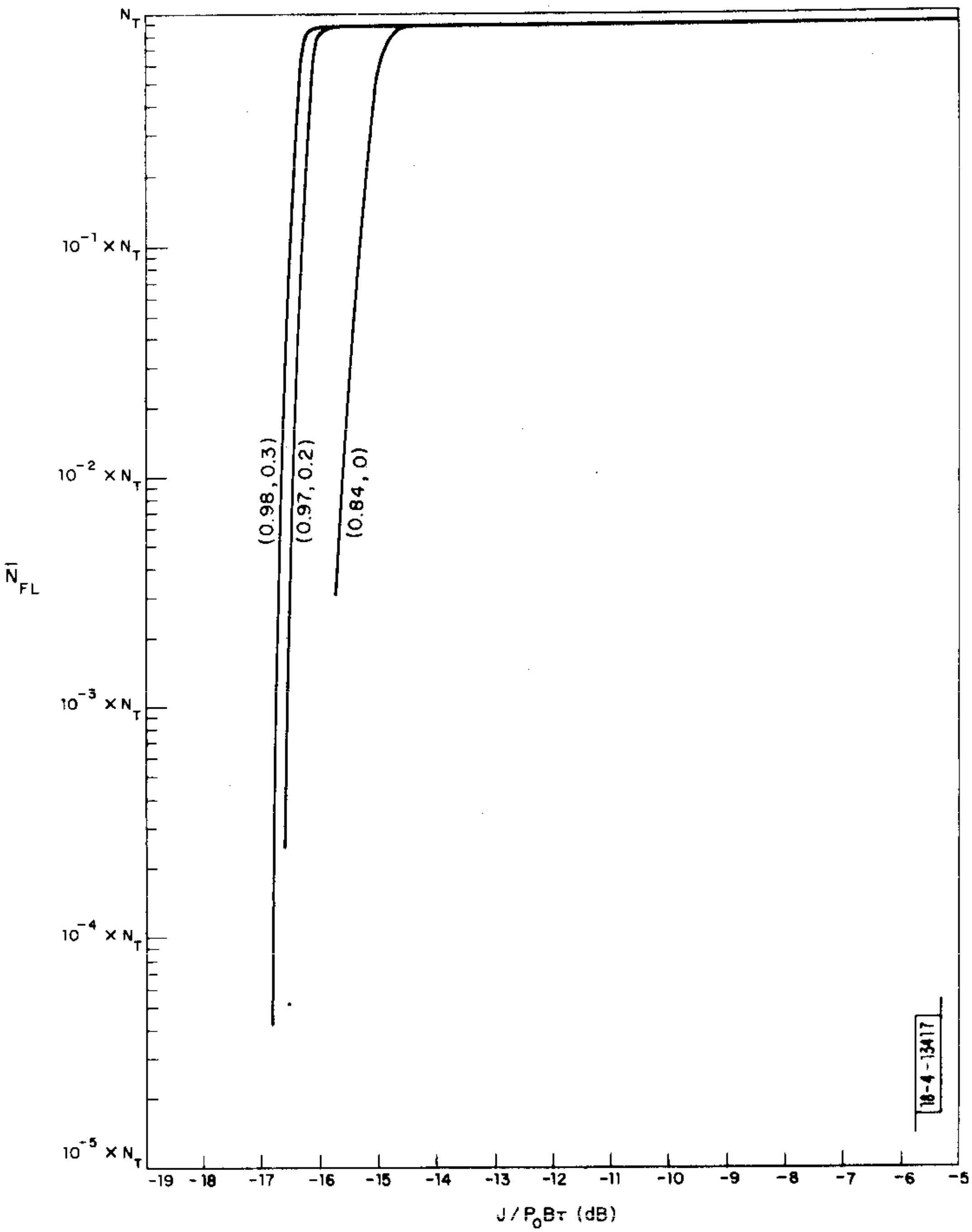

Fig. 2.5. $\overline{N}_{FL}$ vs $\frac{J}{P_o B\tau}$ when $K = 10$, $P_{DL} = 0.999$, for various (p_d, t) combinations.

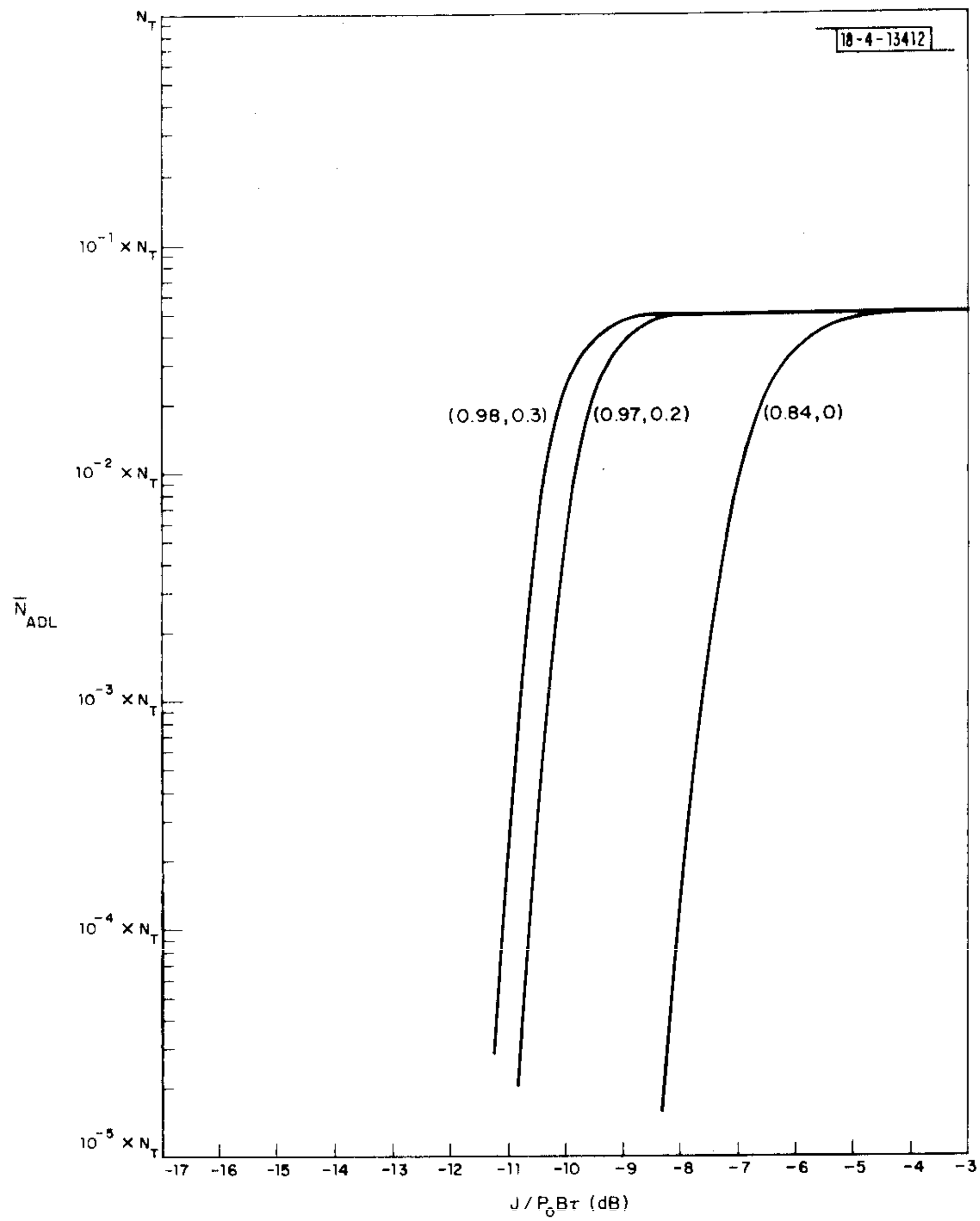

Fig. 2.6. $\overline{N}_{ADL}$ vs $\frac{J}{P_oB\tau}$ when $K = 10$, $P_{DL} = 0.999$, for various (p_d, t) combinations.

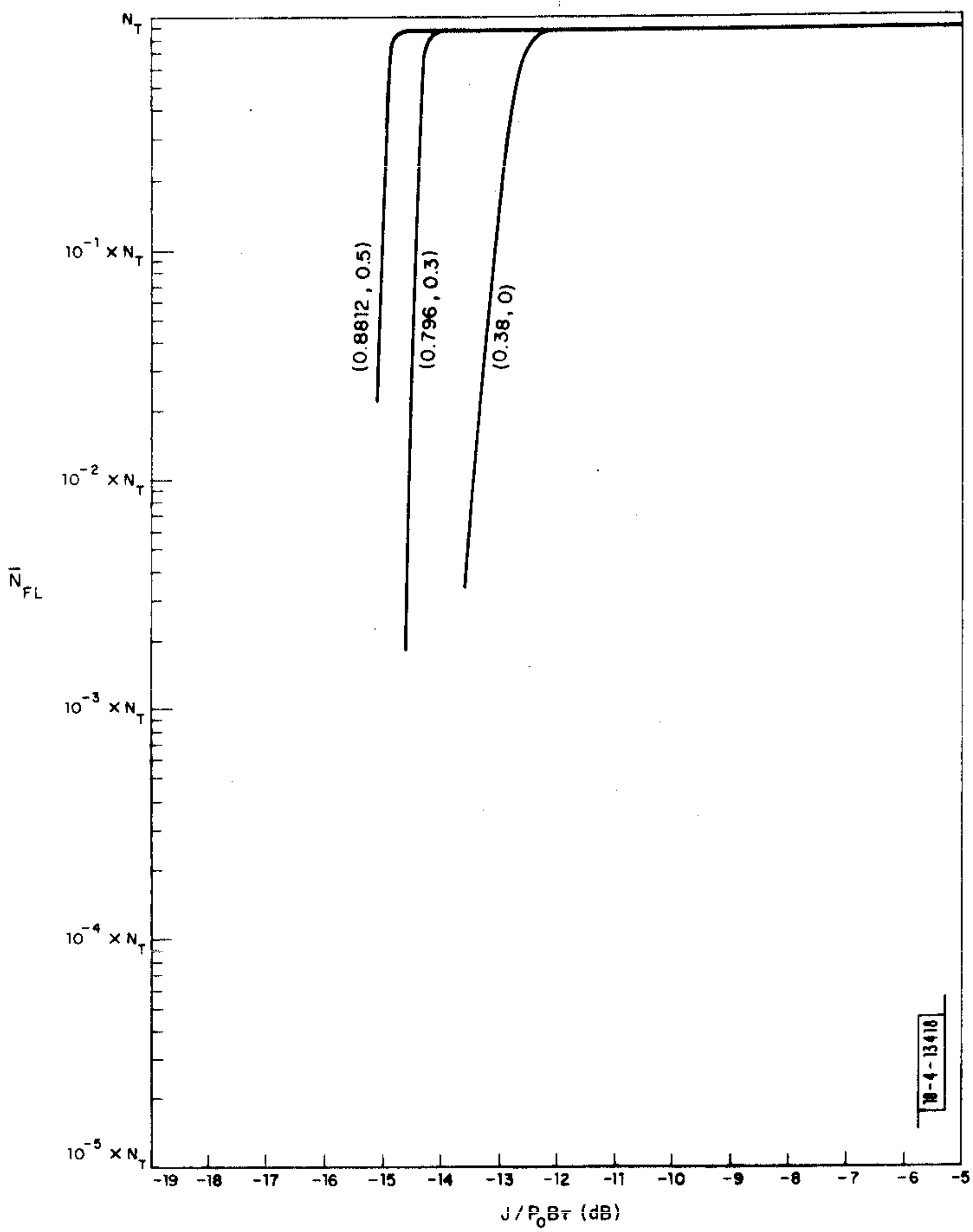

Fig. 2.7. $\overline{N}_{FL}$ vs $\frac{J}{P_o B\tau}$ when $K = 100$, $P_{DL} = 0.88$, for various (p_d, t) combinations.

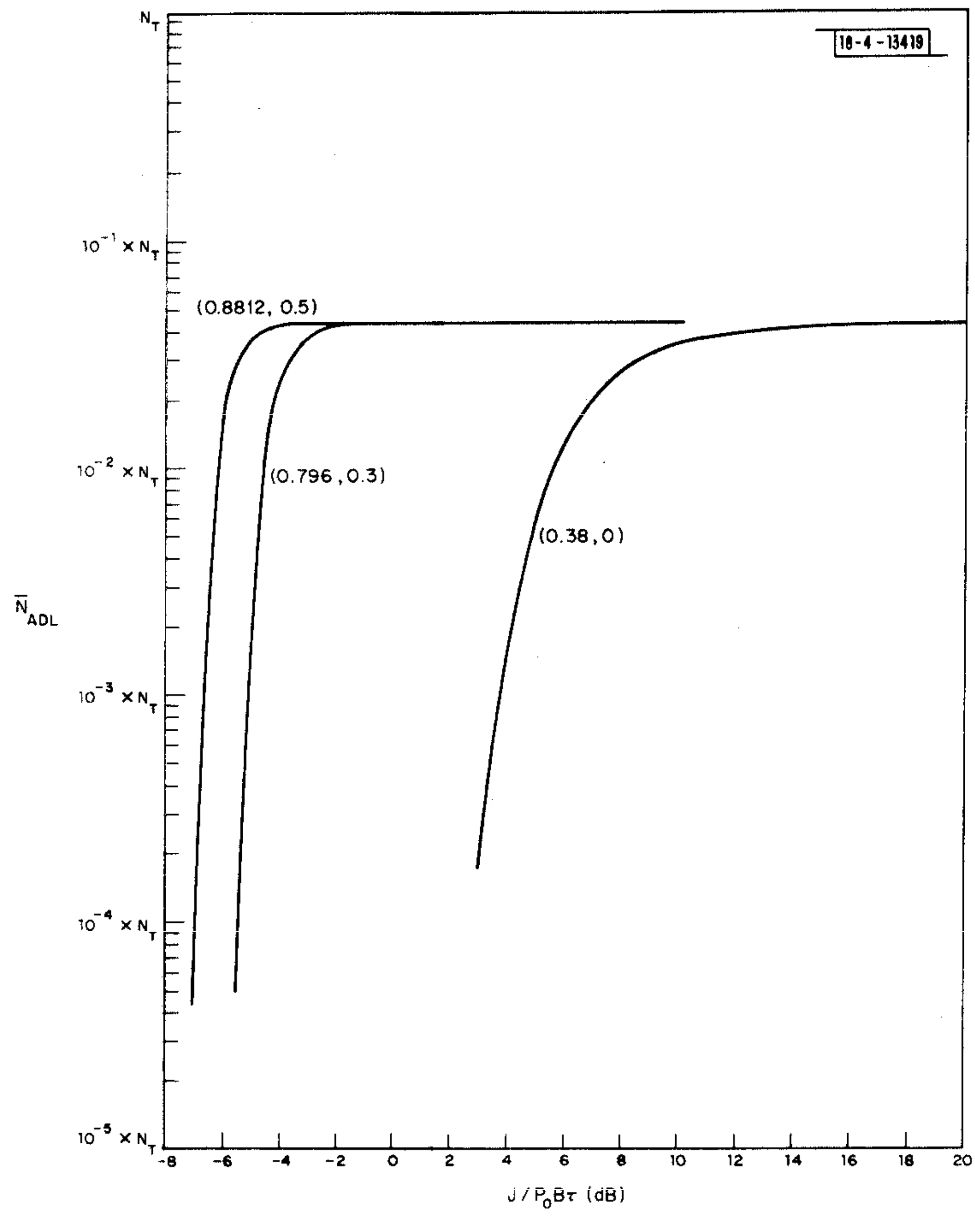

Fig. 2.8. $\overline{N}_{ADL}$ vs $\frac{J}{P_o B\tau}$ when $K = 100$, $P_{DL} = 0.88$, for various (p_d, t) combinations.

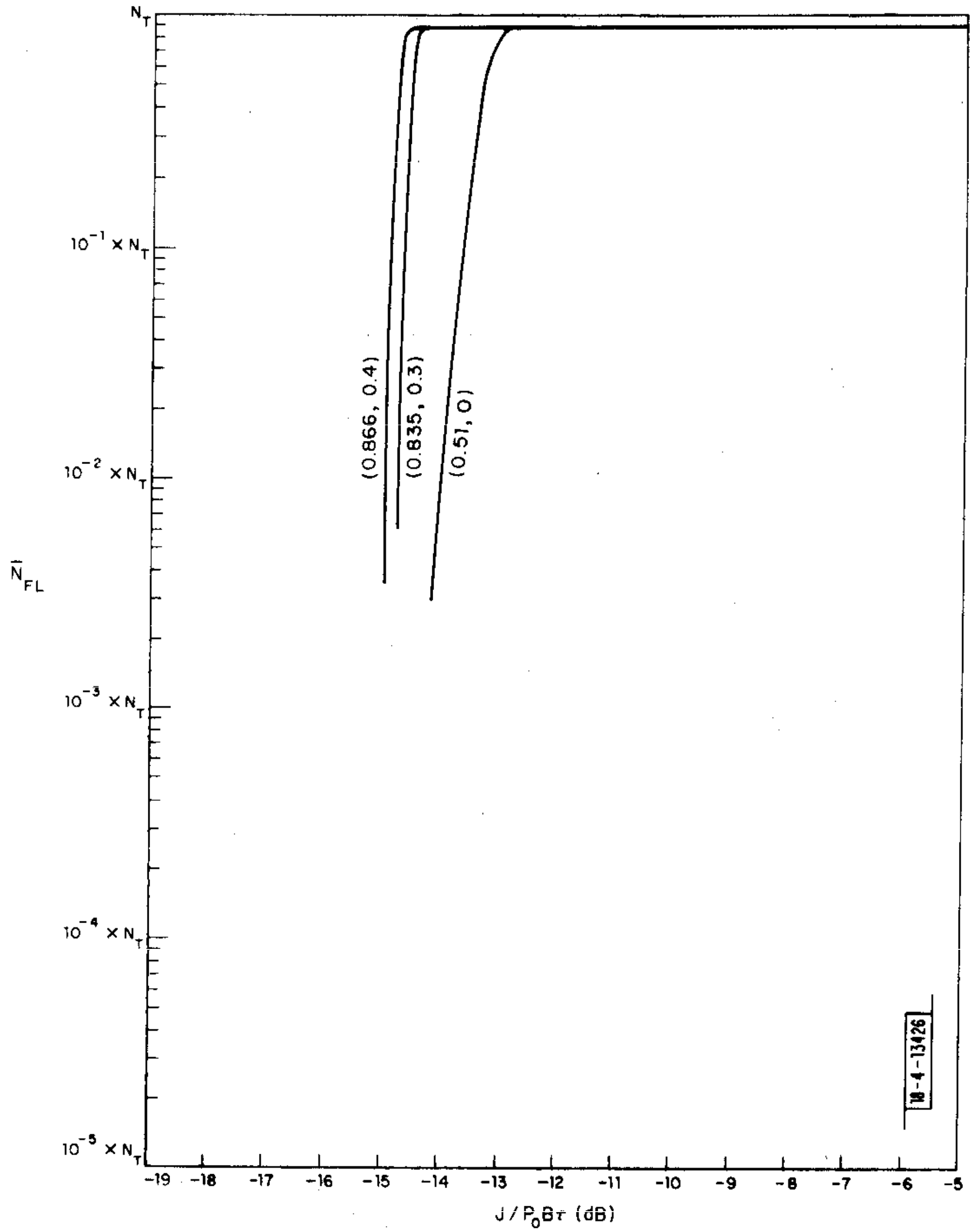

Fig. 2.9. $\overline{N}_{FL}$ vs $\frac{J}{P_o B\tau}$ when $K=100$, $P_{DL}=0.999$, for various (p_d, t) combinations.

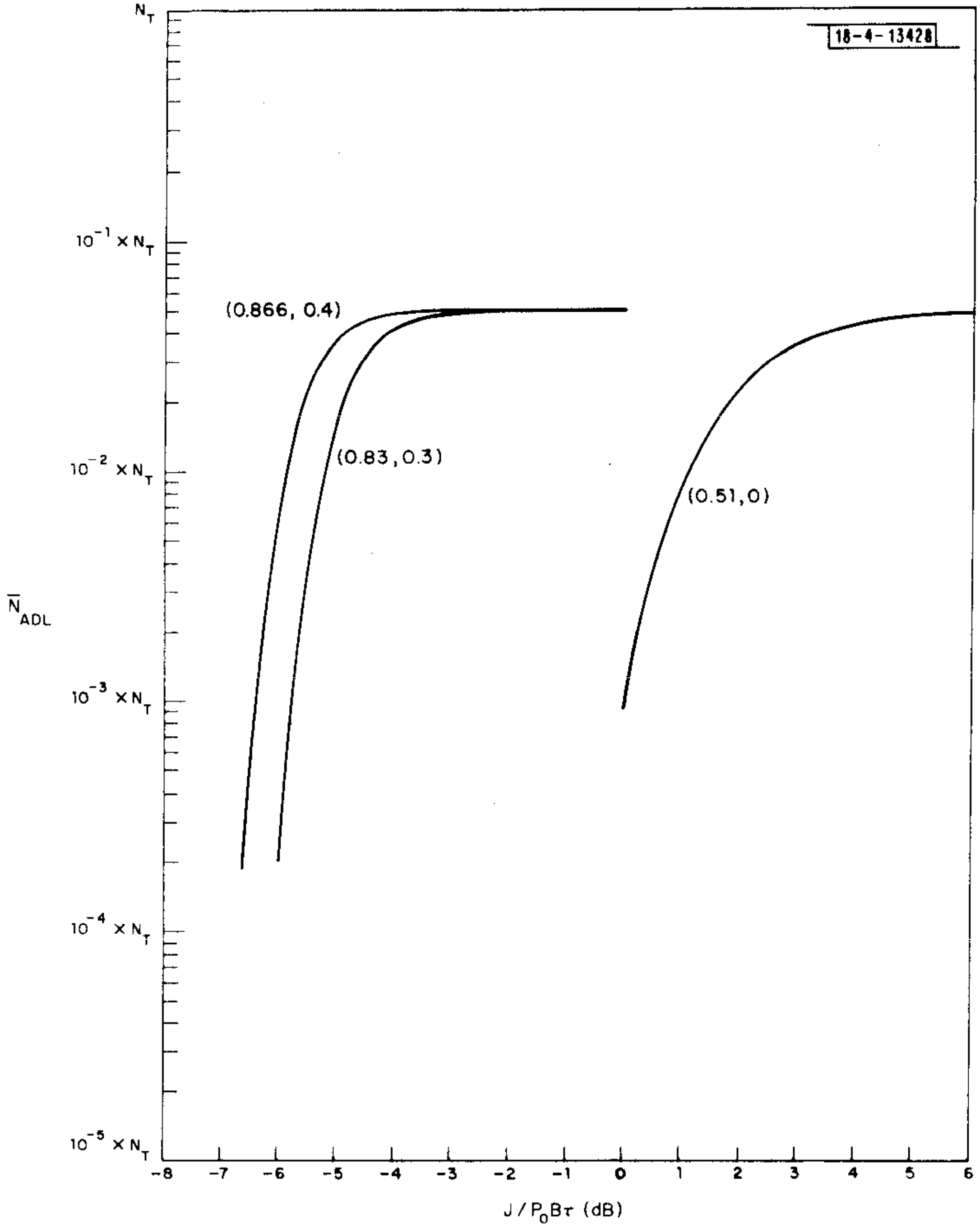

Fig. 2.10. $\overline{N}_{ADL}$ vs $\frac{J}{P_o B\tau}$ when $K = 100$, $P_{DL} = 0.999$, for various (p_d, t) combinations.

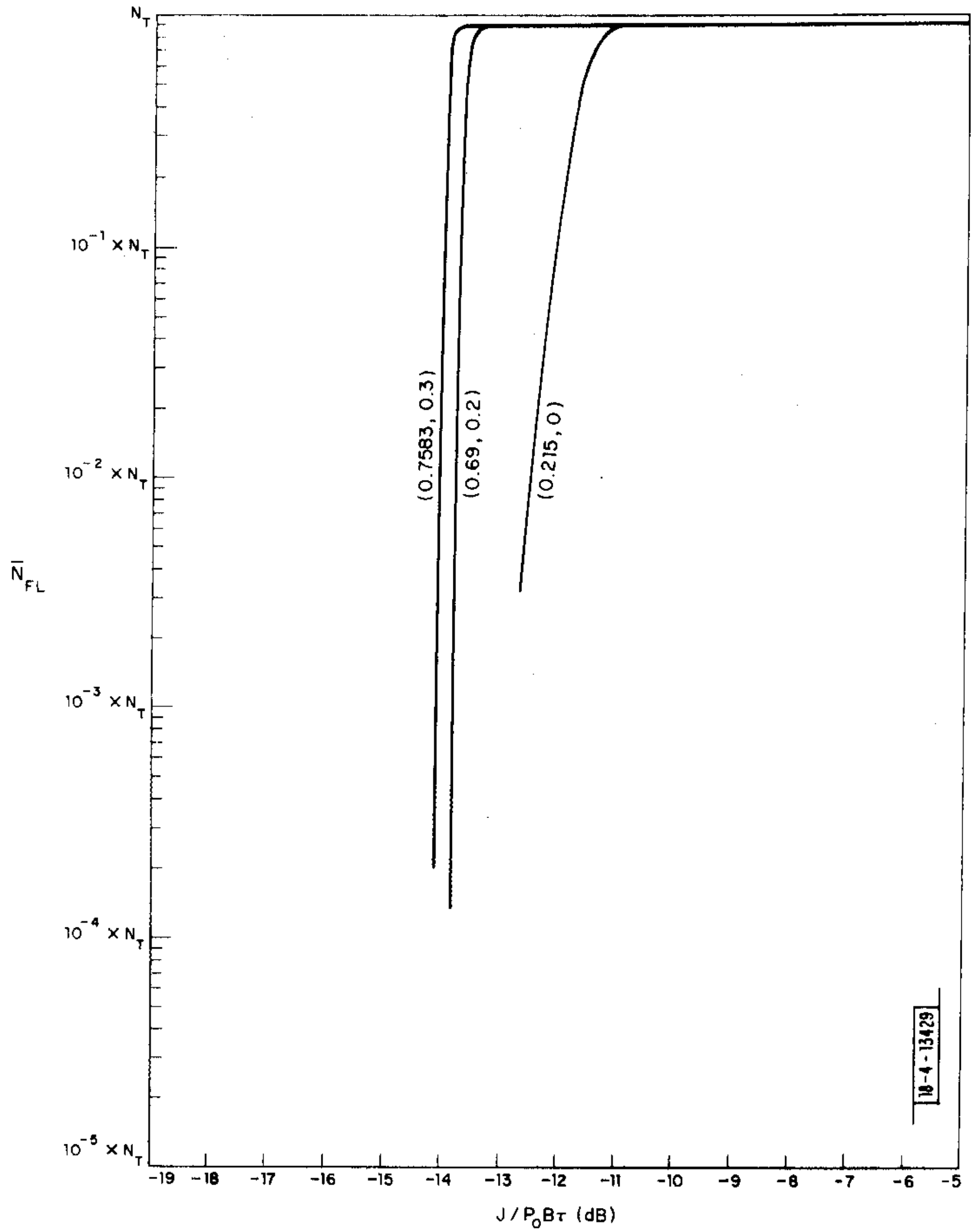

Fig. 2.11. $\bar{N}_{FL}$ vs $\frac{J}{P_o B\tau}$ when $K = 1000$, $P_{DL} = 0.88$, for various (p_d, t) combinations.

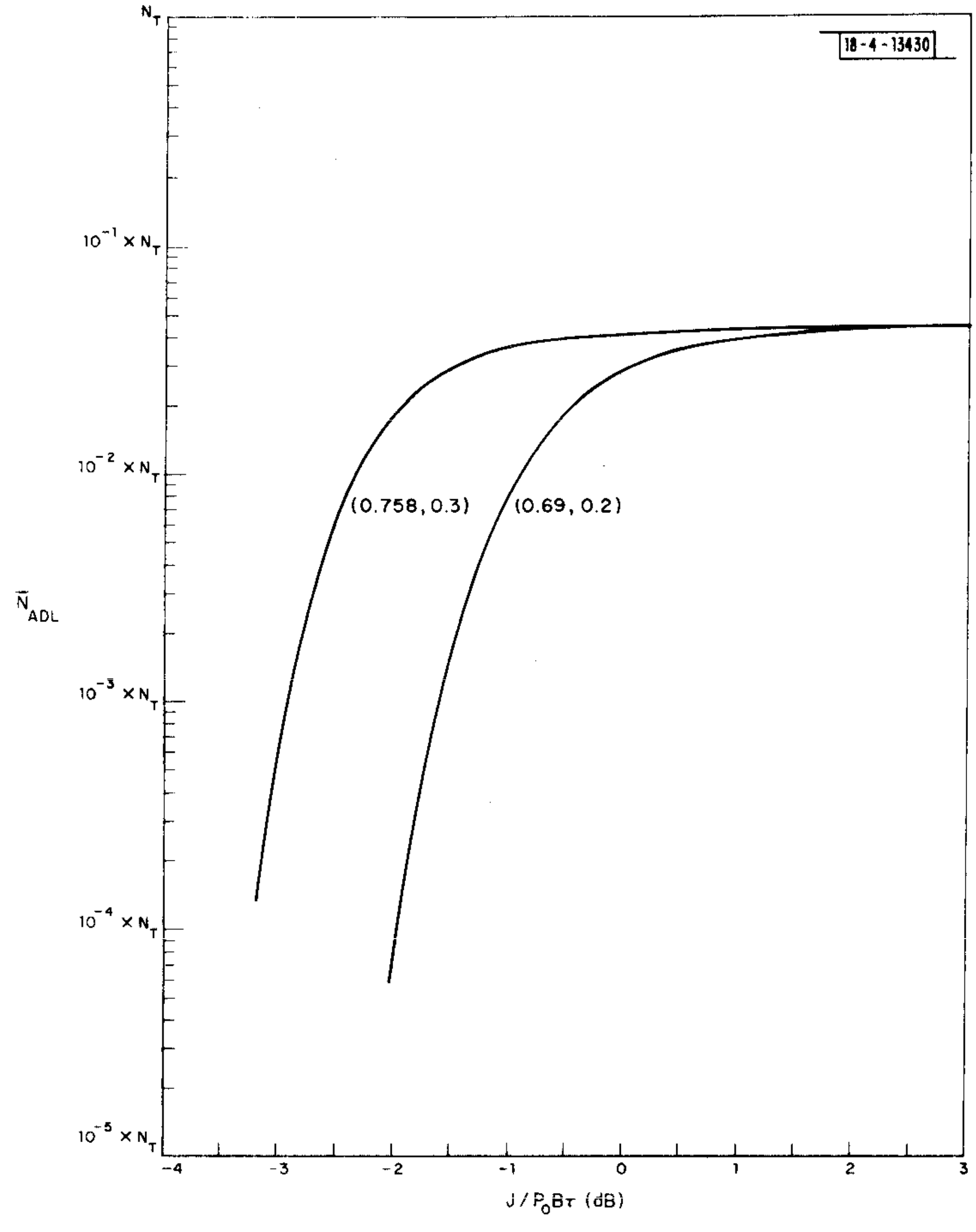

Fig. 2.12. $\overline{N}_{ADL}$ vs $\frac{J}{P_o B\tau}$ when $K = 1000$, $P_{DL} = 0.88$, for various (p_d, t) combinations.

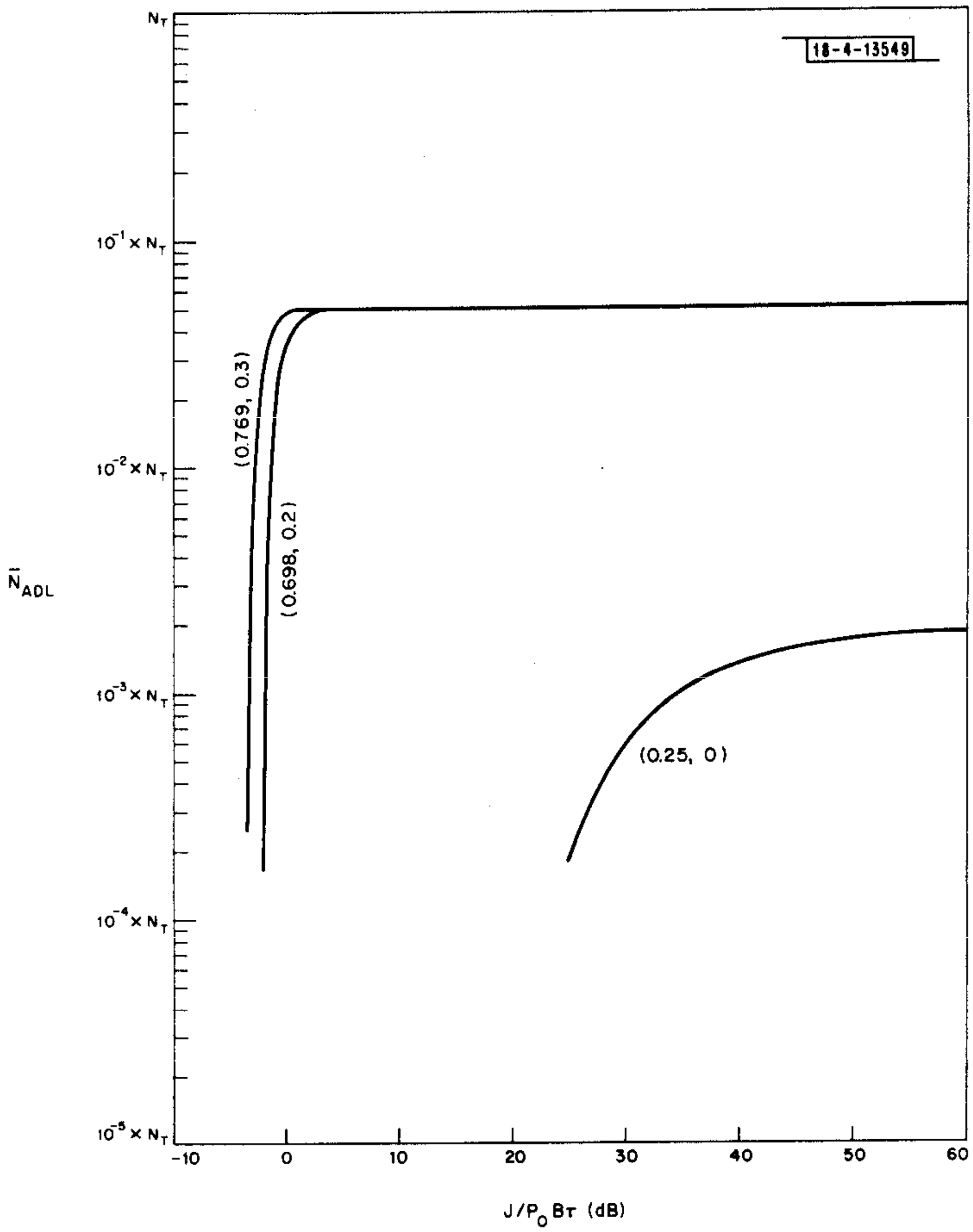

Fig. 2.13. $\overline{N}_{ADL}$ vs $\frac{J}{P_o B\tau}$ when $K = 1000$, $P_{DL} = 0.98$, for various (p_d, t) combinations.

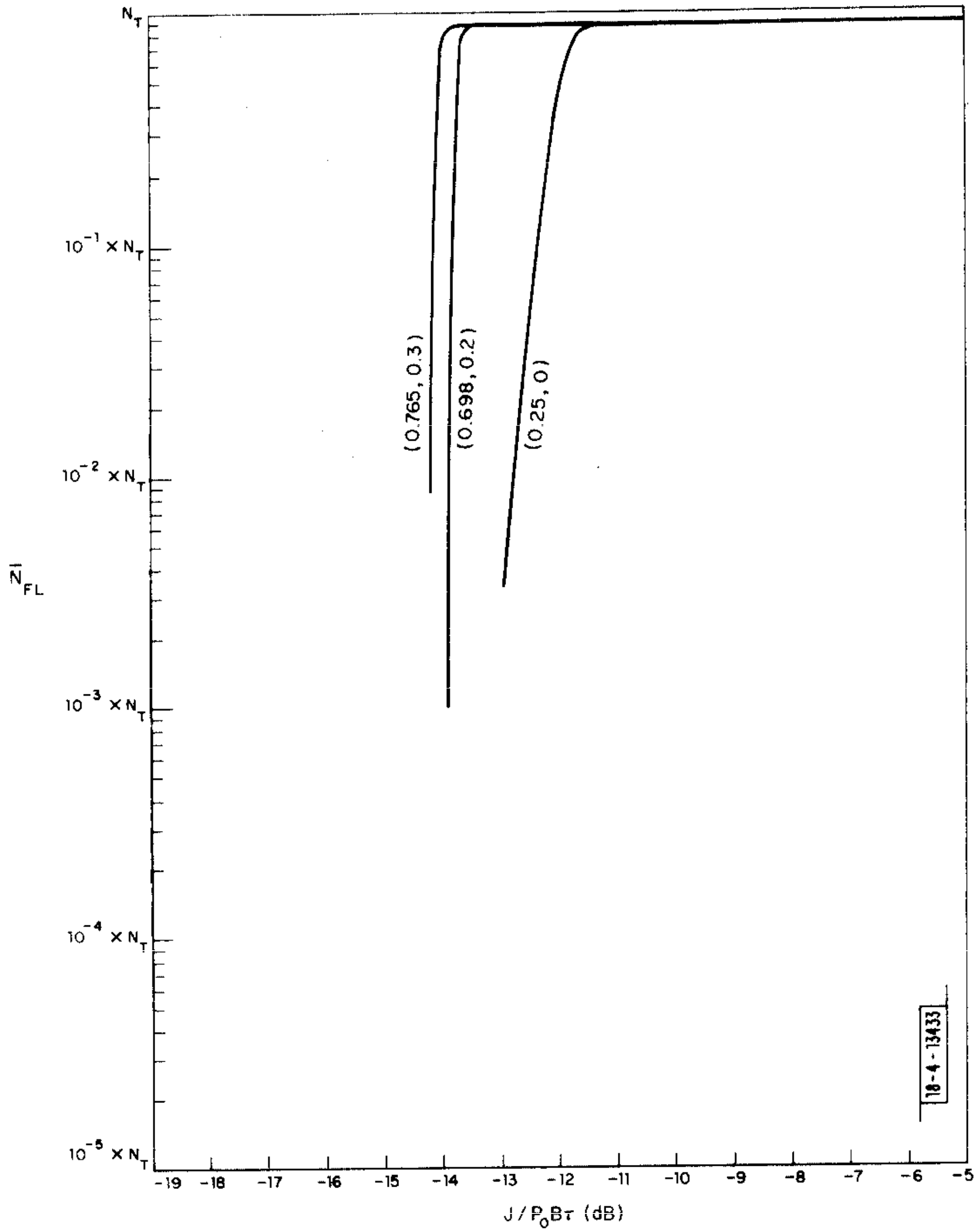

Fig. 2.14. $\overline{N}_{FL}$ vs $\frac{J}{P_o B\tau}$ when $K = 1000$, $P_{DL} = 0.98$, for various (p_d, t) combinations.

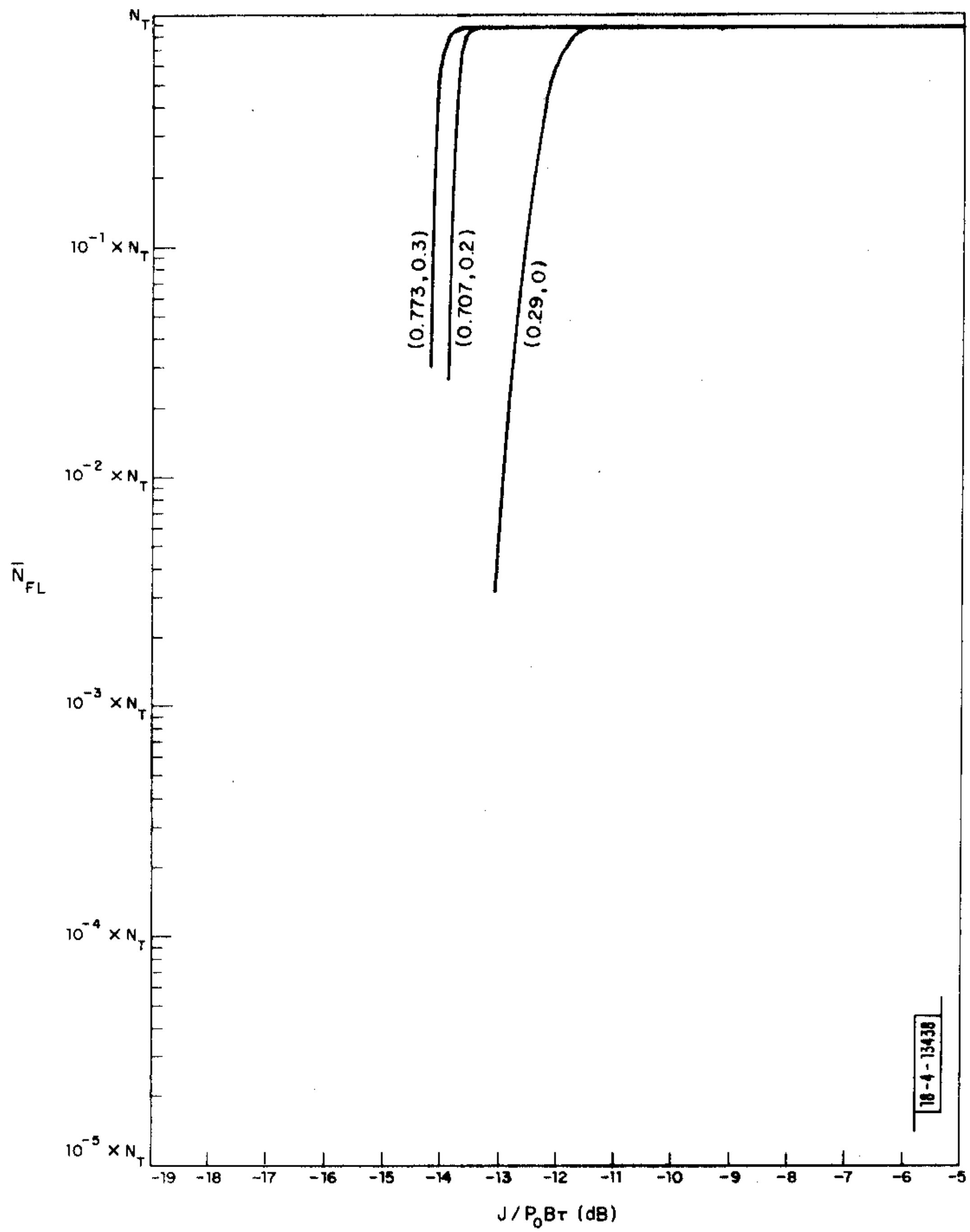

Fig. 2.15. $\overline{N}_{FL}$ vs $\frac{J}{P_oB\tau}$ when $K = 1000$, $P_{DL} = 0.999$, for various (p_d, t) combinations.

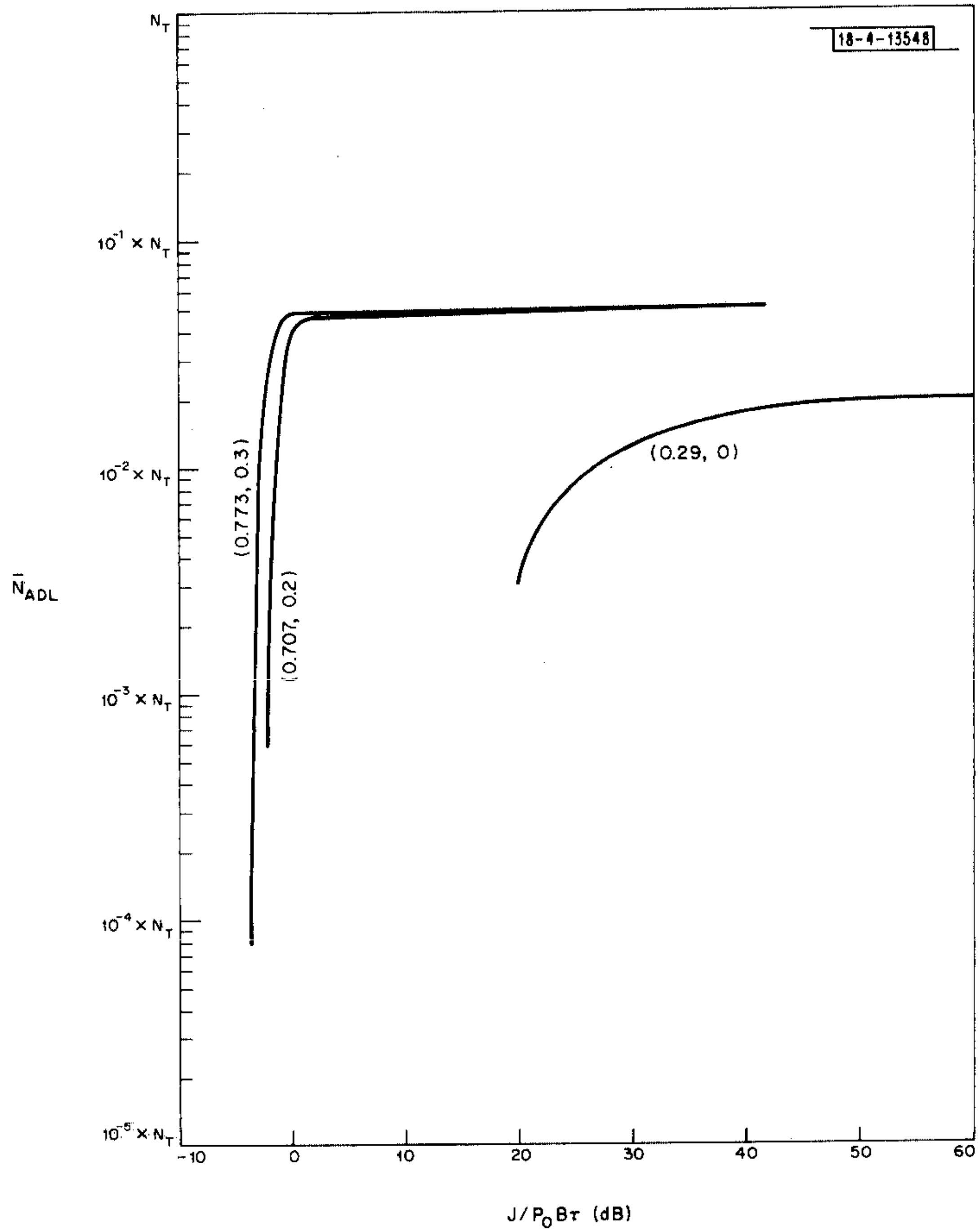

Fig. 2.16. $\bar{N}_{ADL}$ vs $\frac{J}{P_o B\tau}$ when $K = 1000$, $P_{DL} = 0.999$, for various (p_d, t) combinations.

$\frac{J}{P_oB\tau}$ is fixed. As t decreases p_d must decrease in order to maintain a fixed P_{DL}. This implies that p_f is decreasing and thus $\overline{N}_{ADL}$ is decreasing. Turning the variation around, if $\overline{N}_{ADL}$ is fixed then as t decreases $\frac{J}{P_oB\tau}$ must increase in order to maintain the same $\overline{N}_{ADL}$. This of course is based upon the obvious fact that $\overline{N}_{ADL}$ must be monotonically increasing with $\frac{J}{P_oB\tau}$. The same argument holds true for the $\overline{N}_{FL}$ vs. $\frac{J}{P_oB\tau}$ curves. However, here there is one additional factor. As "t" decreases the probability of an aircraft false alarm (on the first stage decision process) will increase. This will offset to some extent the decrease in "p_f" due to a low "t" (at fixed $\frac{J}{P_oB\tau}$) and thus make the $\overline{N}_{FL}$ vs. $\frac{J}{P_oB\tau}$ curve less sensitive to decreasing "t". This is in fact evident when one observes the $\overline{N}_{FL}$ vs. $\frac{J}{P_oB\tau}$ curves and compares them to the $\overline{N}_{ADL}$ vs. $\frac{J}{P_oB\tau}$ curves.

Since it is more difficult for the jammer to cause system degradation at lower values of t than at higher values of t, we can determine the maximum $\frac{J}{P_oB\tau}$ needed in order to make the representative surveillance system unusable, by just considering the curves corresponding to "t=0" for a fixed K and P_{DL}. From Figures 2.3 through 2.16 it is clear that the bound on the average number of false alarms, $\overline{N}_{FL}$ degrades more rapidly with $\frac{J}{P_oB\tau}$ than does the bound on the average number of ambiguous detections. Since each false alarm should be processed by the second stage decision processor, the number of false alarms affects the required complexity of the second stage processor. If, for example, this processor can't handle the number of false alarms than the detection probability might suffer. Since, however, our purpose is to overbound the required jammer, we shall assume that the processor can handle the large number of false alarms and we shall direct our attention to the average number of ambiguous detections.

The maximum required jamming power is summarized in Table 2.8.1. it takes about 10 db more power to disable a K = 10 system than a K = 1 system and about 10 db more power to disable a K = 100 system. In all cases the required jamming power is modest. For K = 1000 the required jamming power depends critically on the desired level of P_{DL}. A desired detection probability of 0.98 or less raises considerable question as to the feasibility of the jammer; however a 2% miss detection probability appears unreasonably large. At the 0.999 level the jammer is certainly feasible although expensive and complex. This level of jamming would require considerable effort and money. Clearly, however, a surveillance system using a tracking length of 1000 is very complex and might, in practice, result in excessive delays.

Table 2.1. Maximum Required Jammer Power Per Satellite for a 10 mJoule Signal Pulse Energy and a 10 MHz Bandwidth.

K	P_{DL}	Jammer ERP (dbw)
10	0.88	46
10	0.999	43
100	0.88	56
100	0.999	51
1000	0.98	> 130
1000	0.999	77

SECTION 3

THE EFFECT OF WAVEFORM MODULATION ON THE PERFORMANCE LOSS DUE TO MULTIPLE ACCESS NOISE

3.1 INTRODUCTION AND SUMMARY OF RESULTS

3.1.1 Problem Definition

An Air-to-Satellite-to-Ground surveillance system operates by having each aircraft transmit a signature in an uncoordinated fashion to a constellation of satellites. Each satellite in the constellation relays the received waveform to a ground station for processing. The position of a given aircraft is computed by first, detecting the signature arrival times, then computing the arrival time differences and applying hyperbolic multilateration.

Multiple access noise arises in an Air-to-Satellite-to-Ground surveillance system in the following way. Because aircraft transmit in an uncoordinated fashion, signatures arrive at any given satellite at random arrival times; Hence, in detecting a given signature the ground processor also encounters unwanted signatures. The net sum of the unwanted signatures received, appears as a random process. This random process is what is termed "multiple access noise."

3.1.2 Assumptions

In carrying out the study of the multiple access noise phenomenon, the following assumptions were made concerning the environment and operation of the Air-to-Satellite-to-Ground surveillance system.

A. Unique Signatures

It was assumed that each aircraft has assigned to it a unique signature which it transmits periodically.

B. Neglect of Other Interference and Losses

Because our main concern was with multiple access noise we assumed absent the presence of any other interference phenomenon; i.e. thermal noise, jammer generated noise. Also, we ignored any losses due to non-ideal link characteristics such as; loss due to a signature being transmitted while an aircraft is banking, loss due to doppler decorrelation. We considered the down-link between satellite and ground station to be perfect.

C. Matched Filter Detector

It was assumed that a matched filter detector was used to detect the arrival time of a given aircraft signature at a ground station. In actual practice, the phase of the received signatures would not be known at the ground station and hence a "matched filter-envelope detector" would be used instead of a "matched filter detector." Of course a receiver which uses "matched filter-envelope detection" will not perform as well as one which uses "matched filter detection." Hence, the performance characteristics which we shall derive will be more optimistic than those which could be obtained with a matched filter envelope detector.

The matched filter detector has its output sampled at a certain rate. If the signal to which the detector is matched is present in the output sample, we assume that it is present at its peak height. Since this is the signal which the matched filter detector is trying to detect our results under this assumption can only be more optimistic than without it. This is due to the fact that we are assuming that when the desired signal is present it is present with its maximum amplitude.

3.1.3 Results

The principal results of our analysis of the multiple access noise interference phenomenon will now be summarized.

A. <u>ROC Comparison</u>

Both upper and lower bounds to the ROC, (Receiver Operating Characteristic), of the matched filter detector were derived. These bounds are asymptotically tight in the sense that they approach the actual ROC as the number of aircraft being considered increases.

B. <u>Gaussian Comparison</u>

Over a large, but restricted, class of signatures and system parameters, we have shown that the ROC of the matched filter detector operating in the presence of multiple access noise is not as preferable as the ROC obtained when the detector operates, only in the presence of additive gaussian noise, with the same signal to interference ratio.

C. <u>PSK Signatures</u>

The output signal to interference ratio in the matched filter detector was computed for the case in which the aircraft signatures were PSK modulated sequences having good auto-correlation properties. The output signal to interference ratio was found to be equal to that obtained when the interference input to the matched filter is a white noise source having a spectral height equal to the average interference power divided by 3 times the reciprocal of the PSK chip duration.

3.2 PROGRAM

It is convenient at this point to describe the program of the remainder of Section 3. We shall study the multiple access noise phenomenon by studying its effect upon the detection of the time of arrival of a given aircraft signature relayed from a given satellite.

In Section 3.3 we shall describe in detail some of the characteristics of the aircraft signatures and of the signal processing used to detect the time of arrival of the aircraft signature of interest at the ground station of interest.

In Section 3.4 we shall describe the statistics of the multiple access noise perturbing the detection process by computing the characteristic function of a multiple access noise sample.

In Section 3.5, upper and lower bounds to the ROC of the matched filter detector are derived. These bounds are evaluated in Section 3.6 assuming typical signature and system parameters. Characteristics of the signature set which optimizes the ROC, over a class of signature sets, are given in Section 3.7.

In Section 3.8 the performance of the matched filter detector operating with interference which is gaussian noise is compared with the performance when the interference is multiple access noise. The validity of assuming the multiple access noise to be gaussian is also studied.

In Section 3.9, the multiple access noise parameters are related to measures of time and bandwidth. The average multiple access noise power is computed when the signatures are PSK modulated sequences having good auto-correlation properties.

3.3 SIGNAL PROCESSING

Let us call the set of aircraft signatures, S. Let $S_j(\cdot)$ be the signature assigned to aircraft "j" the signatures have duration τ and unit energy, i.e.

$$\int_0^\tau S_j^2\,(t)\,dt = 1 \qquad (3.3.1)$$

for j=1,2...N where N is the number of aircraft

Once every α seconds aircraft "j" supplies $S_j(\cdot)$ to a transmitter, which thereupon transmits the bandpass signal $Z_j(\cdot)$. Specifically, if $\hat{t}_j$ is a time at which the transmitter is supplied signature S_j then the following signal is transmitted

$$Z_j(t - \hat{t}_j) = S_j(t - \hat{t}_j)\cos(2\pi f_o(t - \hat{t}_j))$$

This signal is transmitted and received by all satellites and retransmitted to a ground station. The received waveform at the ground processing station from any particular satellite is of the form

$$Z(t) = \sum_{j=1}^{N} Z_j\,(t - t_j)$$

$$Z(t) = S_i(t - t_i)\cos\,(2\pi f_o(t - t_i)) \qquad (3.3.2)$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{N} S_j\,(t - t_j)\cos\,(2\pi f_o(t - t_j))$$

This of course neglects losses in transmitted signal energy due to; range, atmosphere, reflection and any change in carrier frequency.

In order to describe the rest of the signal processing we shall refer to Figure 3.1. This illustrates the processing used to detect the arrival time of aircraft "i's" signature-signal, $Z(\cdot)$, i.e. t_i. Observe the component parts of this figure. After being received at the ground station, $Z(\cdot)$, is immediately mixed with $2 \cos 2\pi f_0 t$. We assume phase synchronization with $Z_i(t - t_i)$. In other words we assume the mixing signal to actually be $2 \cos 2\pi f_0 (t-t_i)$. The resultant signal is then passed through a lowpass filter whose cutoff frequency is far below $2f_0$, but greater than the largest frequency of any of the signatures in S. The resulting output is supplied to a filter matched to aircraft "i's" signature, the signature whose arrival we desire to detect. This filter, Matched Filter "i" has impulse response h(t)

$$h(t) = S_i(-t + \tau) \tag{3.3.3}$$

With $Z(\cdot)$, given by (3.3.2) at the input to the ground station the following signal, y(t), will be generated at the output of Matched Filter "i"

$$y_i(t) = R_{ii}(t,t_i) + n_i(t) \tag{3.3.4}$$

where

$$R_{ii}(t,t_i) = \int_{-\infty}^{\infty} S_i(x-t_i)\, S_i(x-t+\tau)\, dx \tag{3.3.5}$$

$$n_i(t) = \sum_{j \neq i}^{N} n_{ij}(t) \tag{3.3.6}$$

$$n_{ij}(t) = \cos(2\pi f_0(t_i - t_j)) \int_{-\infty}^{\infty} S_j(x-t_j)\, S_i(x-t+\tau)\, dx \tag{3.3.7}$$

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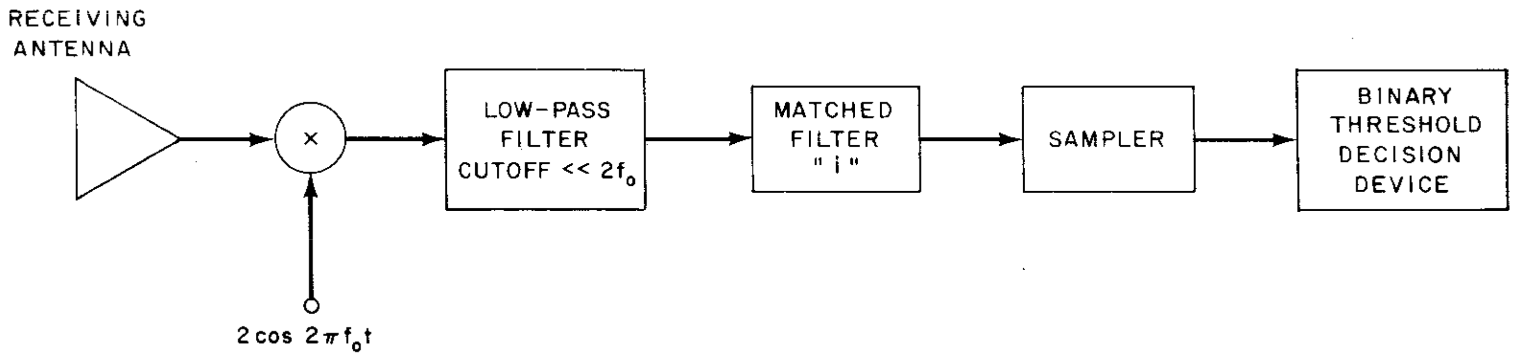

Fig. 3.1. Receiver structure for the detector matched to the signature aircraft "i".

The quantity $R_{ii}(t,t_i)$ in (3.3.4) is the signal term. One should note that because of (3.3.1) this has a peak value of 1. The quantity $n_i(t)$ is the multiple access noise term. The arrival times, t_j, are random and uniformly distributed over each repetition period of length α seconds.

The output of Matched Filter "i", $y_i(t)$, is sampled. Each sample is supplied to the Binary Threshold Decision Device which is the second part of the matched filter detector. If "t" is a sampling time, this decision device operates in the following manner.

If

$y_i(t) \geq \Theta$; it decides that $Z_i(\cdot)$ was received at the ground station at time $t-\tau$

$y_i(t) < \Theta$; it decides that $Z_i(\cdot)$ was not received at the ground station at time $t-\tau$

where Θ is a fixed number between 0 and 1.

Consider the matched filter output, $y_i(t)$ given by (3.3.4). Depending upon t, the signal term, $R_{ii}(t,t_i)$ may be either present in the output or not (i.e., have a zero value). We shall make the following simplifying assumption. If $R_{ii}(t,t_i)$ is present in the output then it is present at its peak value of 1.

Again consider $y_i(t)$. Note that $n_{ij}(t)$ is the contribution to the multiple access noise in the output of Matched Filter "i", due to the processing of aircraft "j'" signature. As is evident $n_{ij}(t)$ can only be nonzero if the reception time t_j satisfies

$$t - 2\tau < t_j < t$$

If t_j is in this interval we define the probability density on the random variable $n_{ij}(t)$ as $C_{ij}(\cdot)$, i.e.,

$$\text{Prob } (n_{ij}(t) \leq a \mid t-2\tau < t_j < t) = \int_{-\infty}^{a} C_{ij}(x)\, dx \quad (3.3.8)$$

We shall refer to $C_{ij}(\cdot)$ as the "amplitude density of $n_{ij}(t)$."

In closing this section we should like to stress that we have assumed a matched filter detector, i.e. the combination of Matched Filter "i" and the Binary Threshold Decision Device. In general this is not the optimum processor. The best linear processor might employ prewhitening filters while the best non-linear processor might attempt to estimate the multiple access noise and then subtract it out.

3.4 STATISTICAL DESCRIPTION OF MULTIPLE ACCESS NOISE

In this section, we shall concern ourselves with describing the multiple access noise statistically. Consider, $n_i(t)$ the multiple access noise sample generated at the output of Matched Filter "i", at time t. We shall compute $M_i(v)$, the characteristic function of $n_i(t)$, and analyze its behavior.

We can proceed directly from the definition of characteristic function obtaining

$$M_i(v) = E\left(e^{jvn_i(t)}\right) \tag{3.4.1}$$

Applying (3.3.6) to (3.4.1) yields

$$M_i(v) = \exp\left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \operatorname{Ln}\left(E\left(e^{jvn_{ij}(t)}\right)\right)\right) \tag{3.4.2}$$

where $\exp(x) = e^x$. In obtaining (3.4.2) we have used the fact that the transmitted aircraft signatures arrive independently at the satellite ground station. This implies that the $n_{ij}(t)$'s are independent.

$E\left(e^{jvn_{ij}(t)}\right)$ can be expanded as follows:

$$E\left(\left(e^{jvn_{ij}(t)}\right)\right) = E\left(e^{jvn_{ij}(t)} \,\middle|\, t-2\tau < t_j < t\right) \operatorname{Prob}\left(t-2\tau < t_j < t\right) + E\left(e^{jvn_{ij}(t)} \,\middle|\, \begin{matrix} t_j < t-2\tau \\ \text{or} \\ t_j > t \end{matrix}\right) \operatorname{Prob}\left(\begin{matrix} t_j < t-t-2\tau \\ \text{or} \\ t_j > t \end{matrix}\right) \tag{3.4.3}$$

t_j of course is the time at which aircraft "j's" signal-signature, $Z_j(\cdot)$ arrives at the ground station.

If, $t_j \leq t-2\tau$ or $t_j \geq t$, then of course $n_{ij}(t) = 0$. We noted this in Section 3.3. The aircraft signatures' arrival times are uniformly distributed in time over the α second repetition period. We use these facts in (3.4.3) to obtain

$$E(e^{jvn_{ij}(t)}) = \frac{2\tau}{\alpha} E\left(e^{jvn_{ij}(t)} \middle| t-2\tau < t_j < t\right) + \left(1 - \frac{2\tau}{\alpha}\right) \tag{3.4.4}$$

Applying the definition of $C_{ij}(\cdot)$, (given by (3.3.8))to (3.4.4) yields

$$E\left(e^{jvn_{ij}(t)}\right) = \left(1 - \frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha} \int_{-\infty}^{\infty} C_{ij}(x)\, e^{jvx}\, dx \tag{3.4.5}$$

Substituting (3.4.5) into (3.4.2) yields the characteristic function $M_i(v)$ as

$$M_i(v) = \exp\left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \text{Ln}\left(\left(1 - \frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha} \int_{-\infty}^{\infty} C_{ij}(x) e^{jvx}\, dx\right)\right) \tag{3.4.6}$$

which is the desired result.

Let n be the generic symbol for $n_i(t)$. Using the properties of characteristic functions one has

$$E(n) = \frac{1}{j} \frac{\partial}{\partial v} M_i(v) \Big|_{v=0} \tag{3.4.7}$$

$$E(n^2) = -\frac{\partial^2}{\partial v^2} M_i(v) \Big|_{v=0} \tag{3.4.8}$$

Applying these formulas to (3.4.6) we have

$$E(n) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{2\tau}{\alpha} \int_{-\infty}^{\infty} x \, C_{ij}(x) \, dx \tag{3.4.9}$$

$$E(n^2) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{2\tau}{\alpha} \int_{-\infty}^{\infty} x^2 \, C_{ij}(x) \, dx \tag{3.4.10}$$

$$+ \left(\frac{2\tau}{\alpha}\right)^2 \left(\sum_{\substack{j=1 \\ j \neq i}}^{N} \int_{-\infty}^{\infty} x \, C_{ij}(x) \, dx \right)^2$$

$$- \left(\frac{2\tau}{\alpha}\right)^2 \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\int_{-\infty}^{\infty} x C_{ij}(x) \, dx \right)^2$$

We may compute $\int_{-\infty}^{\infty} x \, C_{ij}(x) \, dx$ from the definition of $C_{ij}(\cdot)$ given by (3.3.8) and the definition of $n_{ij}(t)$ given by (3.3.7). Computing this we have

$$\int_{-\infty}^{\infty} x \;\; C_{ij}(x)\, dx = E\,(n_{ij}(t))$$

$$= E\,[\cos\,(2\pi f_o(t_i - t_j)) \int_{-\infty}^{\infty} S_j(x-t_j) \;\; S_i(x-t+\tau)\, dx]$$

$$\int_{-\infty}^{\infty} x\, C_{ij}(x)\, dx = \; E\,[\cos\,(2\pi f_o t_i)\, \cos\,(2\pi f_o t_j) \int_{-\infty}^{\infty} S_j(x-t_j) S_i(x-t+\tau)]$$

$$+ E\,[\sin\,(2\pi f_o t_i)\, \sin\,(2\pi f_o t_j) \int_{-\infty}^{\infty} S_j(x-t_j) S_i(x-t+\tau)]$$

t_i and t_j are independent hence,

$$\int_{-\infty}^{\infty} x C_{ij}(x)\, dx = E\,[\cos\,(2\pi f_o t_i)]\; E\,[\cos\,(2\pi f_o t_j) \int_{-\infty}^{\infty} S_j(x-t_j) S_i(x-t+\tau)]$$

$$+ E\,[\sin\,(2\pi f_o t_i)]\; E\,[\sin\,(2\pi f_o t_j) \int_{-\infty}^{\infty} S_j(x-t_j) S\;(x-t+\tau)] \qquad (3.4.11)$$

t_i is uniformly distributed over the α second transmission period. We shall assume that α is an integral number of periods of the carrier, (i.e. $f_o\alpha$ is an integer) then

$$E\,[\cos\,(2\pi f_0 t_i)] \;=\; E\,[\sin\,(2\pi f_0 t_i)] \;=\; 0$$

Applying this to (3.4.11) we have

$$\int_{-\infty}^{\infty} x\; C_{ij}(x)\; dx = 0 \qquad (3.4.12)$$

Applying (3.4.12) to (3.4.9) and (3.4.10) brings

$$E(n) \;=\; 0$$

$$E(n^2) = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{2\tau}{\alpha} \int_{-\infty}^{\infty} x^2\; C_{ij}(x)\; dx \qquad (3.4.13)$$

$E(n)$ of course is the mean of the multiple access noise sample. $E(n^2)$ is the average power of the multiple access noise sample.

We conclude this section with the following theorem which is proven in Appendix E. It deals with the asymptotic character of $M_i(v)$.

<u>Theorem 3.1</u>

If the signature set, S, is such that there is a common amplitude density, i.e.

$$C_{ij}(x) \;=\; C(x) \quad \text{for all } j \neq i$$

and if τ/α is fixed, then $M_i(v)$ is asymptotically (with N) the characteristic function of a gaussian random variable having mean "0" and variance $\frac{N2\tau}{\alpha}\int_{-\infty}^{\infty} x^2\; C(x)\; dx$.

3.5 BOUNDS TO THE RECEIVER OPERATING CHARACTERISTIC

The presence of multiple access noise at the output of Matched Filter "i" affects the operation of the matched filter detector. The ROC (receiver operating characteristic) is the usual way in which the performance of the matched filter detector is measured. In this part we shall define the ROC as the curve p_f as a function of p_d where

$$p_f = \text{Prob}\,(y_i(t) \geq \theta \mid R_{ii}(t,t_i) = 0) \tag{3.5.1}$$

$$P_d = \text{Prob}\,(y_i(t) \geq \theta \mid R_{ii}(t,t_i) = 1) \tag{3.5.2}$$

t is a sampling time. $y_i(t)$ is the output of Matched Filter "i" at time t. It is given by (3.3.4). $R_{ii}(t,t_i)$ is the signal portion of $y_i(t)$. "$R_{ii}(t,t_i)=0$" implies signal absent on output, "$R_{ii}(t,t_i) = 1$" implies signal present on output.

In this section we shall compute upper and lower bounds to the ROC. The upper bound is generated from an upper bound to p_f and a lower bound to p_d and represents the worst case ROC. The lower bound is generated from a lower bound to p_f and an upper bound to p_d and represents a best case ROC. Before computing the bounds it will be convenient to describe once again the operation of the binary threshold decision device. It operates in the following way:

If $y_i(t) \geq \theta$; it decides $Z_i(\cdot)$ was received at the ground station at time $t-\tau$

(3.5.3)

If $y_i(t) < \theta$; it decides $Z_i(\cdot)$ was not received at the ground station at time $t-\tau$

3.5.1 Upper Bound to the ROC

A. Upper Bound to p_f

From the 3.5.1 and 3.3.6 it follows that

$$p_f = \text{Prob}\left(\sum_{\substack{j=1\\ j\neq i}}^{N} n_{ij}(t) \geq \theta\right) \tag{3.5.4}$$

Using Chernoff Bounds we can upper bound (3.5.4) as

$$p_f \leq \exp\left[-\rho\theta + \sum_{\substack{j=1\\ j\neq i}}^{N} \text{Ln}(Ee^{\rho n_{ij}(t)})\right] \tag{3.5.5}$$

where $\rho \geq 0$, but is otherwise arbitrary.

Now, we can evaluate $E\,(e^{\rho n_{ij}(t)})$ very easily by just letting v equal to ρ/j in (3.4.5). This results in

$$E(e^{\rho n_{ij}(t)}) = (1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x} C_{ij}(x)dx \tag{3.5.6}$$

Substituted into (3.5.5) yields

$$p_f \leq \exp\left[-\rho\theta + \sum_{\substack{j=1\\ j\neq i}}^{N} \text{Ln}\left((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x} C_{ij}(x)dx\right)\right] \tag{3.5.7}$$

for arbitrary positive ρ. We now insist that ρ be picked as that positive ρ which minimizes the right hand side of (3.5.7). This results in the desired bound.

B. Lower Bound to p_d

From 3.5.2 and 3.3.6 it follows that

$$p_d = 1\text{-Prob}\left(\sum_{\substack{j=1\\ j\neq i}}^{N} -n_{ij}(t) \geq (1-\theta)\right) \tag{3.5.8}$$

Using Chernoff Bounds, we have the following upper bound:

$$\text{Prob}\left(\sum_{\substack{j=1\\ j\neq i}}^{N} -n_{ij}(t) \geq (1-\theta)\right) \leq \exp\left[-\gamma(1-\theta) + \sum_{\substack{j=1\\ j\neq i}}^{N} \text{Ln}(E\, e^{-\gamma n_{ij}(t)})\right] \tag{3.5.9}$$

where $\gamma \geq 0$, but is otherwise arbitrary

Now, we can evaluate $E(e^{-\gamma n_{ij}(t)})$ very easily by just letting v equal to $-\gamma/j$ in (3.4.5). This results in

$$E(e^{-\gamma n_{ij}(t)}) = (1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{-\gamma x} C_{ij}(x)dx \tag{3.5.10}$$

Substituting into (3.5.9) yields

$$\text{Prob}\left(\sum_{\substack{j=1\\j\neq i}}^{N} -n_{ij}(t) \geq (1-\theta)\right) \leq \exp\left[-\gamma(1-\theta) + \sum_{\substack{j=1\\j\neq i}}^{N} \text{Ln}\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{-\gamma x} C_{ij}(x)dx\right)\right] \tag{3.5.11}$$

Hence,

$$p_d \geq 1-\exp\left[-\gamma(1-\theta) + \sum_{\substack{j=1\\j\neq i}}^{N} \text{Ln}\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{-\gamma x} C_{ij}(x)dx\right)\right] \tag{3.5.12}$$

We now insist that γ be picked as that positive number which maximizes the right hand side of (3.5.12). This yields the desired bound. (3.5.7) and (3.5.12) constitute an upper bound to the ROC, we shall state them together for convenience

$$p_f \leq \exp\left[-\rho\theta + \sum_{\substack{j=1\\j\neq i}}^{N} \text{Ln}\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x} C_{ij}(x)dx\right)\right] \tag{3.5.13}$$

$$p_d \geq 1-\exp\left[-\gamma(1-\theta) + \sum_{\substack{j=1\\j\neq i}}^{N} \text{Ln}\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{-\gamma x} C_{ij}(x)dx\right)\right]$$

ρ is picked to minimize the upper bound to p_f. γ is picked to maximize the lower bound to p_d. ρ,γ are restricted to be positive. These two bounds together constitute an upper bound to the ROC.

Before deriving the lower bound to the ROC let us stop and discuss the ROC upper bound given by (3.5.13). Notice that this bound depends upon the multiple access noise through four parameters; N, τ, α and $\{C_{ij}(x)\}$. As N increases and/or (τ/α) increases the upper bound to p_f increases and the lower bound to p_d decreases. This indicates a degeneration of the performance of the matched filter detector. This, of course, is logical since as N and/or (τ/α) increase the average number of a mismatched signatures contributing to the amplitude of an output sample of Matched Filter 'i' increases. This increases the probability of multiple access noise having a large magnitude on the sample, thus causing the performance of the threshold device to deteriorate. The amplitude densities $\{C_{ij}(x)\}$ are functions of the detailed structure of the signatures in the signature set, S. In a sense, they represent the modulation scheme which S itself represents. In system design, as we should see, one should choose an S having a $\{C_{ij}(x)\}$ which causes the upper bound to p_f to be small and the lower bound to p_d to be large.

3.5.2 Lower Bound to the ROC

A lower bound to the ROC is developed by obtaining a lower bound to p_f and an upper bound to p_d. In order to obtain the bound to p_f and the bound to p_d, one merely applies a Chernoff lower bound to the right hand side of (3.5.4) and another Chernoff lower bound to the term Prob($\sum -n_{ij}(t) \geq (1-\theta)$) in (3.5.8). Both of these tasks are just straight forward applications of the bound given by Gallager[3]. We shall just state the result

$$p_f \geq \frac{1}{2} \exp[-\rho\theta + (\sum_{\substack{j=1 \\ j \neq i}}^{N} ((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha} \int_{-\infty}^{\infty} e^{\rho x} C_{ij}(x)dx)) - \phi_1]$$

$$P_d \leq 1-\frac{1}{2}\exp[-\gamma(1-\theta) + (\sum_{\substack{j=1 \\ j\neq i}}^{N} Ln((\frac{1-2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{-\gamma x} C_{ij}(x)dx)) - \phi_2] \tag{3.5.14}$$

ρ is the same positive ρ chosen to optimize (3.5.7). γ is the same positive value chosen to optimize (3.5.12).

$$\phi_1 = -2\rho_1 \sqrt{2\frac{\partial^2}{\partial r^2}(\sum_{\substack{j=1 \\ j\neq i}}^{N} Ln((\frac{1-2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x} C_{ij}(x)dx)\Big|_{r=\rho_1})}$$

where ρ_1 is the solution, r_1, of

$$\theta = \frac{\partial}{\partial r}(\sum_{\substack{j=1 \\ j\neq i}}^{N} Ln((\frac{1-2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x} C_{ij}(x)dx))$$

$$-\sqrt{2\frac{\partial^2}{\partial r^2}(\sum_{\substack{j=1 \\ j\neq i}}^{N} Ln((\frac{1-2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{rx} C_{ij}(x)dx))}$$

$$\phi_2 = -2\gamma_1 \sqrt{2\frac{\partial^2}{\partial g^2}(\sum_{\substack{j=1 \\ j\neq i}}^{N} Ln((\frac{1-2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{-gx} C_{ij}(x)dx\Big|_{g=\gamma_1})}$$

where γ_1 is the solution, g, of

$$(1-\theta) = \frac{\partial}{\partial g}\left(\sum_{\substack{j=1 \\ j\neq i}}^{N} Ln\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{-gx} C_{ij}(x)dx\right)\right)$$

$$- \sqrt{2\frac{\partial^2}{\partial g^2}\left(\sum_{\substack{j=1 \\ j\neq i}}^{N} Ln\left(\left(1-\frac{2\tau}{\alpha}\right)+ \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{-gx} C_{ij}(x)dx\right)\right)}$$

The bounds given by (3.5.14) constitute the lower bound to the ROC.

Comparing the upper bound to the ROC given by (3.5.13) with the lower bound given by (3.5.14), one may note that they are asymptotically (with N) tight.

3.6 RECEIVER OPERATING CHARACTERISTIC COMPARISON

In the previous section, we computed upper and lower bounds to the ROC. These are given by (3.5.13) and (3.5.14) respectively. As is evident from observing these bounds they are functions of the signature set parameters; N, τ/α, and $\{C_{ij}(x)\}$. In this section we shall compute these bounds for a typical value set of (N, τ/α) and for several different candidate $\{C_{ij}(x)\}$'s. The amplitude density set, $\{C_{ij}(x)\}$, depends upon the detailed structure of the signature set and hence, in a sense, represents the modulation design of the signature set. By comparing bounds to the ROC for different $\{C_{ij}(x)\}$'s we are equivalently comparing ROC's for different signature set designs.

We fix N and τ/α at the following values:

(i) $N=10^5$

(ii) $\tau/\alpha=2(10^{-5})$

and we consider four different amplitude density sets; $\{C_{ij}(x)\}_1$, $\{C_{ij}(x)\}_2$, $\{C_{ij}(x)\}_3$, and $\{C_{ij}(x)\}_4$. Each of these amplitude density sets has one common density, i.e.,

$$C_{ij}(x) = C_1(x) \text{ for all } C_{ij}(x) \in \{C_{ij}(x)\}_1$$
$$C_{ij}(x) = C_2(x) \text{ for all } C_{ij}(x) \in \{C_{ij}(x)\}_2$$
$$C_{ij}(x) = C_3(x) \text{ for all } C_{ij}(x) \in \{C_{ij}(x)\}_3$$
$$C_{ij}(x) = C_4(x) \text{ for all } C_{ij}(x) \in \{C_{ij}(x)\}_4$$

In addition, we constrain the second moments of each of the common densities to be equal to b, i.e.,

$$\int_{-\infty}^{\infty} x^2 C_r(x)dx = b \qquad r=1,\ldots 4$$

The common densities; $C_1(x),\ldots C_4(x)$, are totally representative of the amplitude density sets; $\{C_{ij}(x)\}_1,\ldots\{C_{ij}(x)\}_4$. We can talk either about one or the other. The common densities that we have chosen to compare are shown in Figure 3.2.

Our constraint on the second moment of the common densities is equivalent to a constraint on the average multiple access noise power. Observe Equation (3.4.13). Constraining the second moments of the amplitude densities to equal b, constrains the average multiple access noise power to equal $\frac{2(N-1)\tau}{\alpha}$ b. Thus, in comparing different common densities for an equal value of second moment, we are comparing different signature designs under the constraint that each will yield the same amount of average multiple access noise power.

The ROC's computed for the fixed values of $(N, \tau/\alpha)$ and for the amplitude densities; $C_1(x)\ldots C_4(x)$, are shown in Figures 3.3 through 3.6. In each figure the solid contour corresponds to the upper bound to the ROC. The broken contour corresponds to a lower bound to the ROC. Each contour corresponds to a constant value of the ratio $1/\sqrt{3b}$, the value of which is labelled on the contour. Each point on the contour corresponds to a different value of θ.

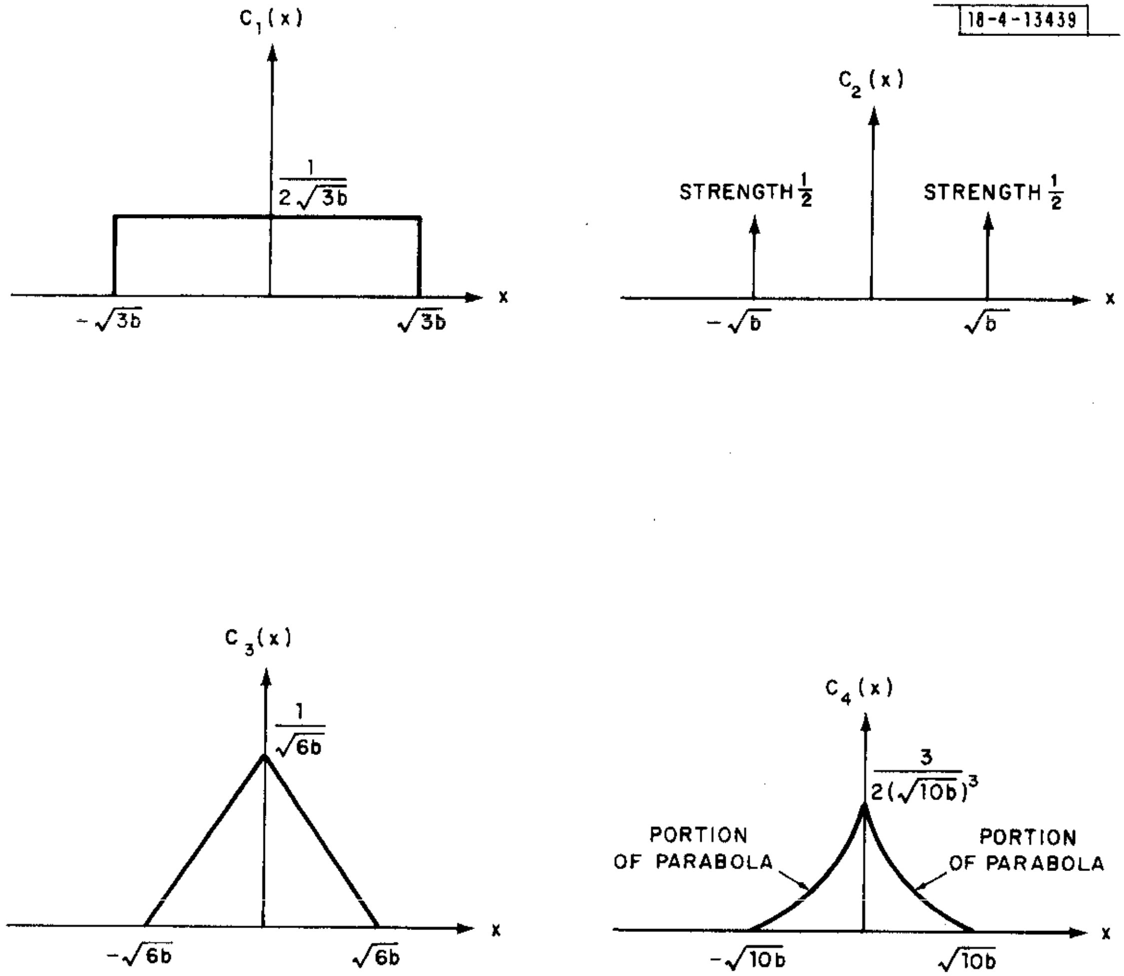

Fig. 3.2. The amplitude densities; $C_1(x)$, $C_2(x)$, $C_3(x)$, $C_4(x)$.

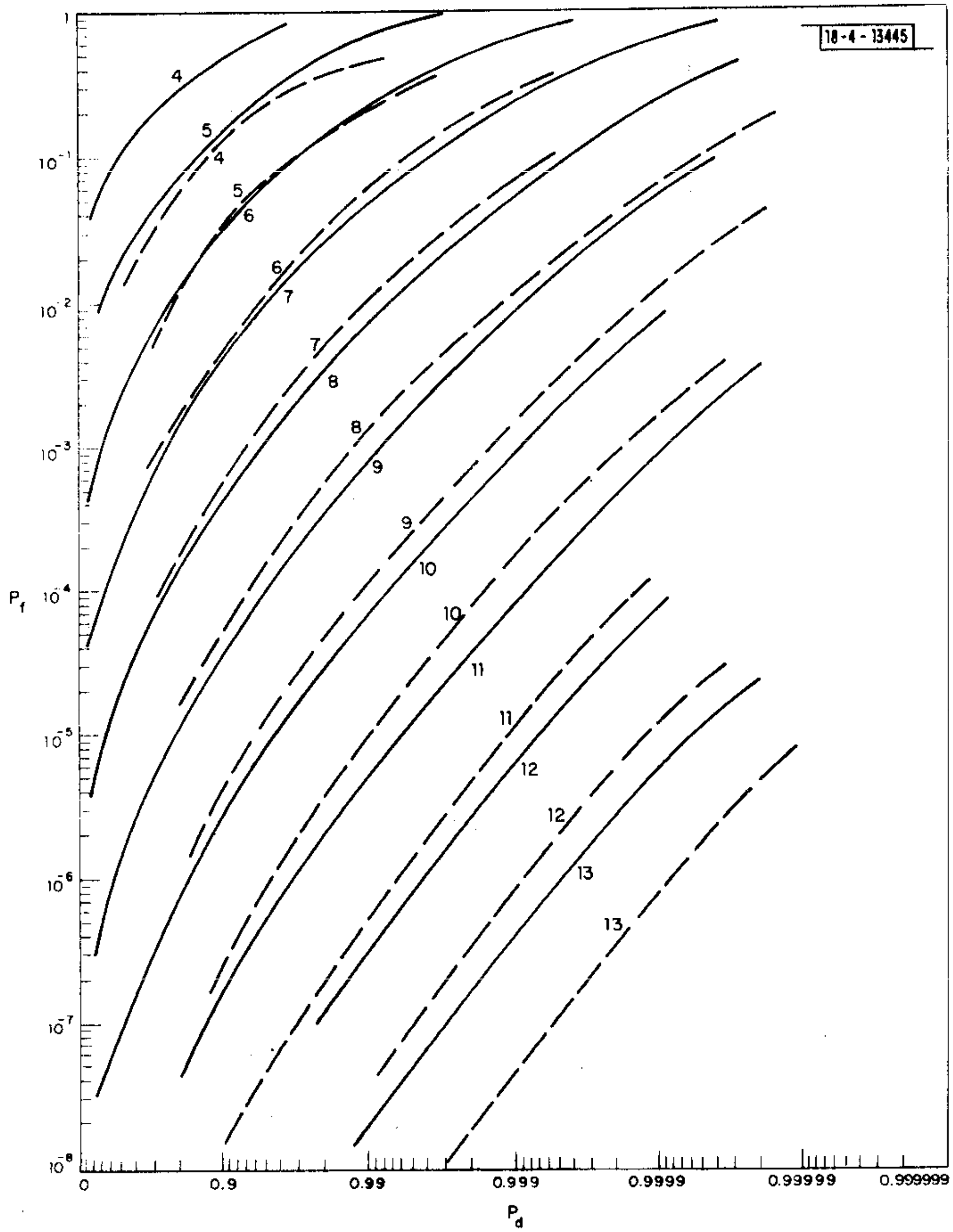

Fig. 3.3. Upper and lower bounds to the ROC when the amplitude density is $C_1(x)$, for various values of $\frac{1}{\sqrt{3b}}$.

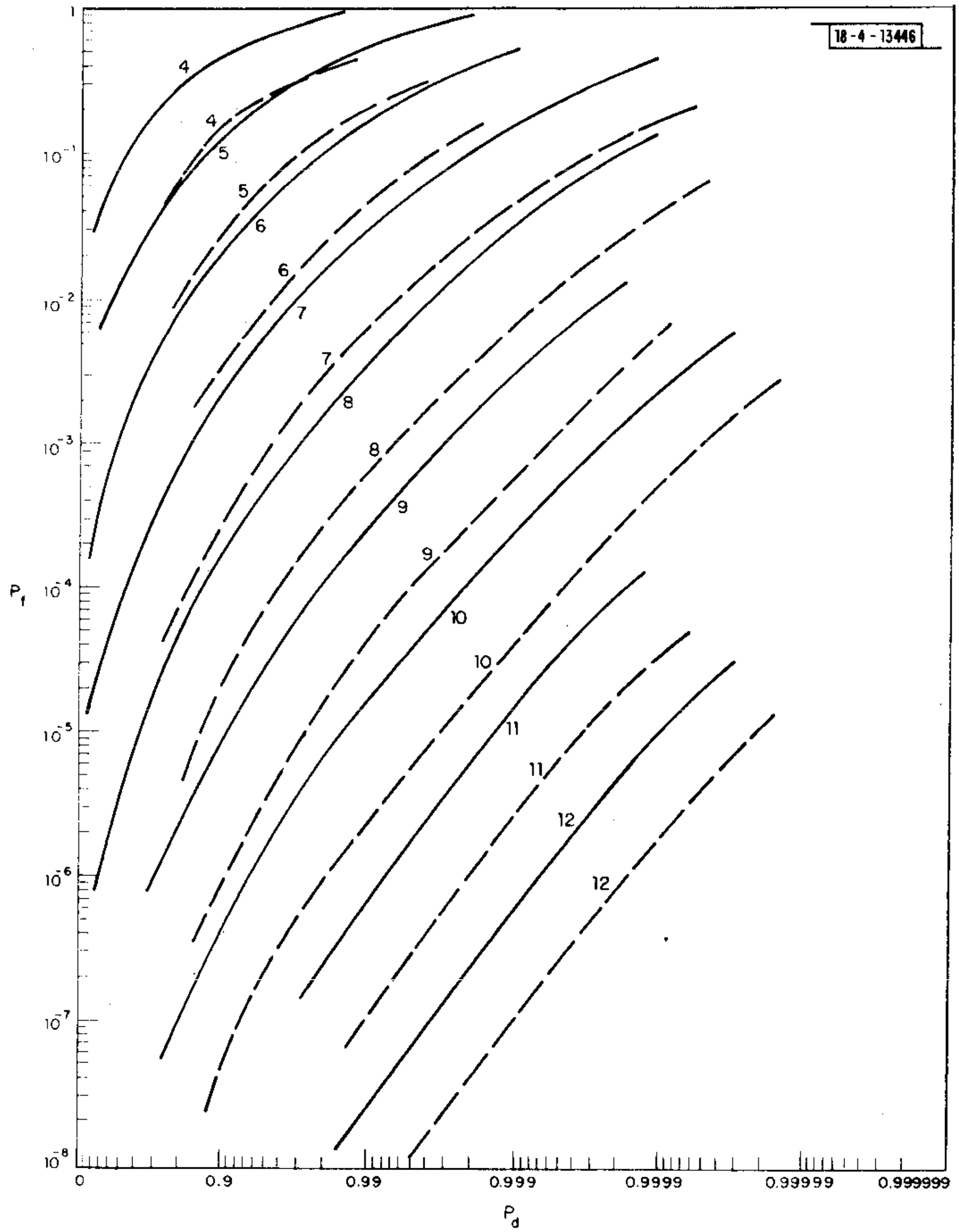

Fig. 3.4. Upper and lower bounds to the ROC when the amplitude density is $C_2(x)$, for various values of $\frac{1}{\sqrt{3b}}$.

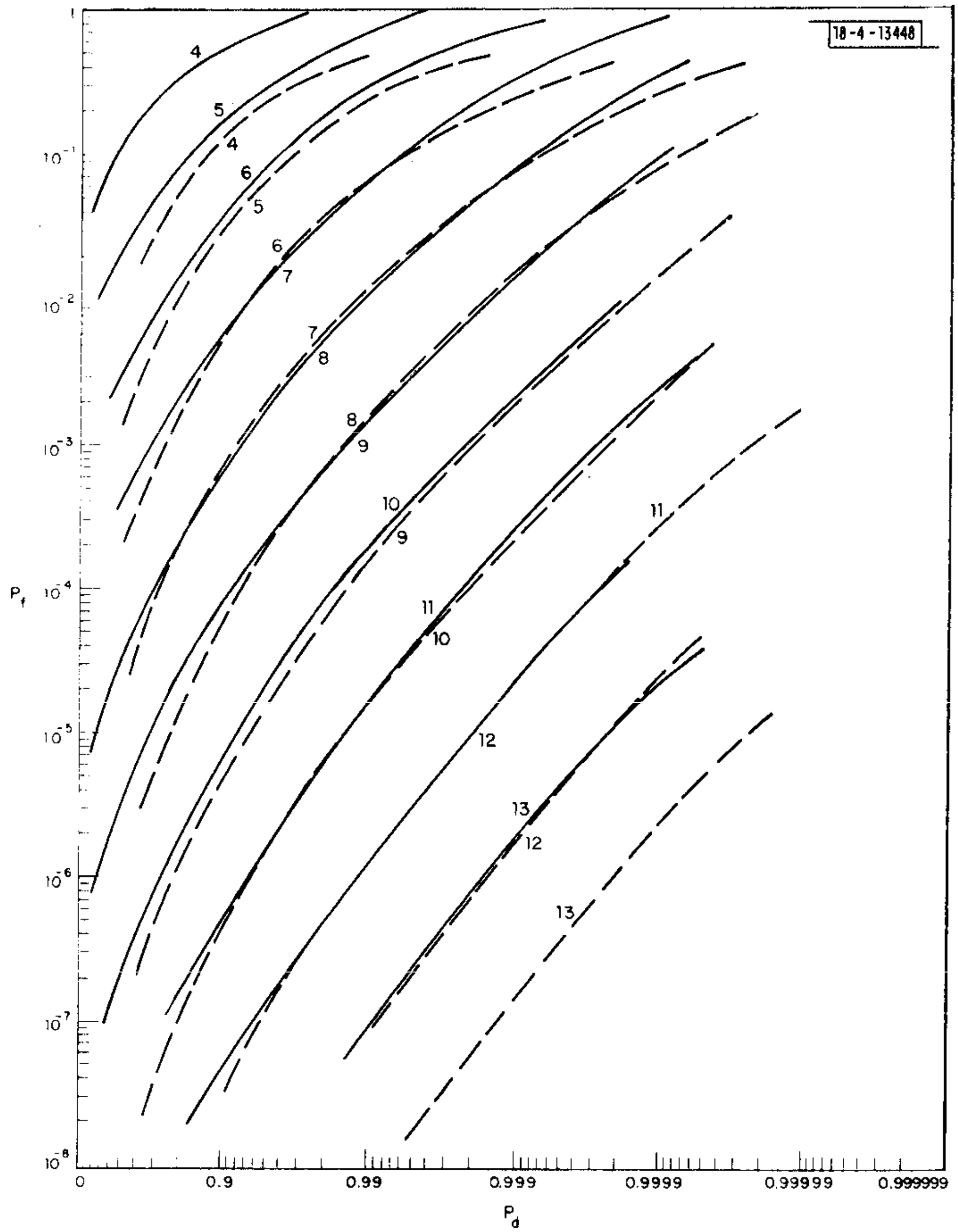

Fig. 3.5. Upper and lower bounds to the ROC when the amplitude density is $C_3(x)$, for various values of $\frac{1}{\sqrt{3b}}$.

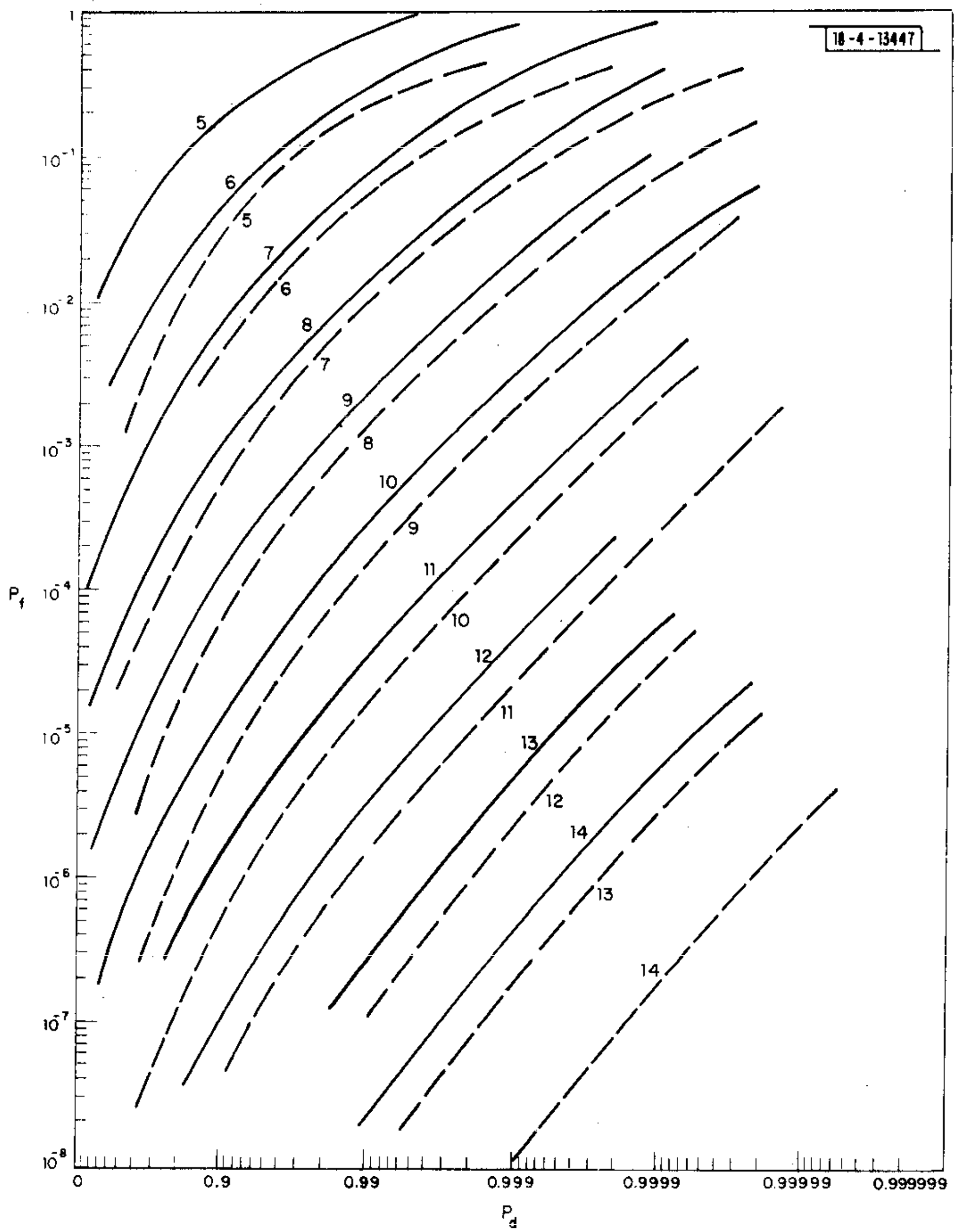

Fig. 3.6. Upper and lower bounds to the ROC when the amplitude density is $C_4(x)$, for various values of $\frac{1}{\sqrt{3b}}$.

We can compare the ROC's shown in Figures 3.3 through 3.6 by comparing contours corresponding to the same value of $\frac{1}{\sqrt{3b}}$.

For purposes of comparison just consider the upper bound contours. It appears that the amplitude density $C_2(x)$ yields the most preferable ROC in that, for a given value of p_d and $\frac{1}{\sqrt{3b}}$, it has the smallest upper bound to p_f (and also the smallest lower bound). After $C_2(x)$, $C_1(x)$ yields an ROC which is more preferable than that of $C_3(x)$ or $C_4(x)$. $C_4(x)$ has the least preferable ROC. We can order the common amplitude densities in terms of their preferability as; $C_2(x)$, $C_1(x)$, $C_3(x)$, $C_4(x)$. Of course, these are only a few of the of possible amplitude density sets. One may, in fact, believe that this order of preferability is strongly a function of the specific values of $(N,\tau/\alpha)$ chosen. We shall deal with this question in the next section where we shall show that $C_2(x)$ is in fact the optimum amplitude density set under the constraints of; fixed average multiple access noise power, and a symmetry condition.

Before concluding this section, let us note that for each ROC set the performance of the matched filter detector deteriorates (i.e., larger p_f's for the same p_d) as $\frac{1}{\sqrt{3b}}$ decreases). Since the average multiple access noise power is $\frac{2(N-1)\tau}{\alpha}$ b, this implies that performance is deteriorating as the average multiple access noise power increases. Of course, this is what one expects intuitively.

3.7 THE QUASI OPTIMAL AMPLITUDE DENSITY

In this section, we consider the problem of finding that $\{C_{ij}(x)\}$, which for, a given value of average multiple access noise power, gives simultaneously the smallest upper bound to p_f and the largest lower bound to p_d, (i.e., the best ROC upper bound) as given by (3.5.13). Theorem 3.2 will be the answer to this problem. In proving the theorem we shall require our signature set, S, to be of a certain type. We shall require it to be a "uniform signature set." A definition of this is first given.

Definition 3.1

S is a uniform signature set if for every $j \neq i$ and x

$$C_{ij}(x) = C_{ij}(-x)$$

We now state the relevant theorem dealing with the best ROC upper bound.

Theorem 3.2: Let:

1. $\mathcal{C}$ be a set $\{C_{ij}(x),\ j=1,\ldots N\ j\neq 1\}$ which could correspond to a uniform signature set (i.e. satisfy the constraint of Definition 3.1.

2. $\mathcal{X}(\mathcal{C})$ be the set of all such C's which have average multiple access noise power fixed at p,

$$E(n^2) = p$$

3. N, τ/α be fixed

Then: The element of $\mathcal{X}(\mathcal{C})$, $\mathcal{C}^*$, which simultaneously maximizes the lower bound to p_d and minimizes the upper bound to p_f over $\mathcal{X}(\mathcal{C})$, (thereby giving the most preferable upper bound to the ROC), is $\mathcal{C}^*$

$$\mathcal{C}^* = \{C_{ij}^*(x),\ j=1,\ldots N, j\neq i\}$$

$$C_{ij}^*(x) = \frac{1}{2} u_o\left(x - \sqrt{\frac{p\alpha}{(N-1)2\tau}}\right) + \frac{1}{2} u_o\left(x + \sqrt{\frac{p\alpha}{(N-1)2\tau}}\right) \quad (3.7.1)$$

for every $j \neq i$

($u_o(x)$ is the unit impulse)

Before embarking upon a proof of this theorem, it will lend some clarity if it is discussed to some extent. Basically, the theorem says that if the average multiple access power is fixed at p then the best upper bound to the ROC is obtained if the amplitude density is common for each "j" and is of the form given by (3.7.1). First of all, the common amplitude density given by (3.7.1) implies that the multiple access noise components, $n_{ij}(t)$, will be two valued, the amplitude can have a value either $+\sqrt{\frac{p\alpha}{(N-1)2\tau}}$ or $-\sqrt{\frac{p\alpha}{(N-1)2\tau}}$. A typical such $n_{ij}(t)$ is illustrated in Figure 3.7. Secondly, the common amplitude density implies that the total multiple access noise is divided equally among all mismatched signatures arriving in the matched filter 'i' detector. Each mismatched signature contributes a quantity of power equal to $\frac{p\alpha}{(N-1)2\tau}$ to the multiple access noise power when it contributes. Finally, one may note that a two valued $n_{ij}(t)$ of amplitude $\sqrt{\frac{p\alpha}{(N-1)2\tau}}$, realizes a multiple access noise power contribution of $(p\alpha/(N-1)2\tau)$ with the smallest possible peak height. Thus, the quasi-optimal $\{C_{ij}(x)\}$ given by (3.7.1) has the characteristic that it spreads the fixed average multiple access noise power over equal contributions from all mismatched signatures. It obtains each equal noise power contribution from an $n_{ij}(t)$ having the smallest

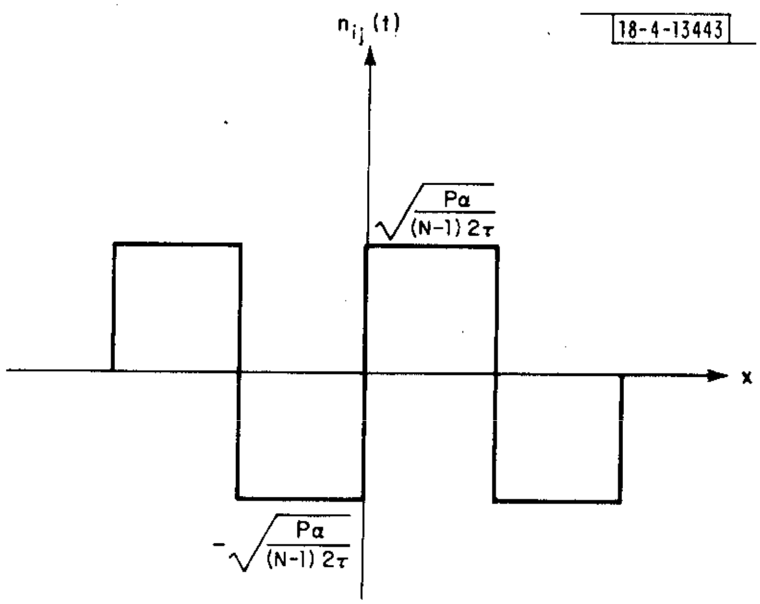

Fig. 3.7. Multiple access noise component implied by the optimal density.

possible peak height. We may conclude from this that it is peak height of the multiple access noise component which dominates degradation in the performance of the matched filter detector.

We shall now prove the theorem. We shall do this by first showing that $\{C_{ij}^{*}(x)\}$ given by (3.7.1) gives the smallest possible upper bound to p_f. Later we shall argue that it gives the greatest lower bound to p_d thus completing the proof.

Proof:

Observe the upper bound to p_f given in (3.5.13)

$$p_f \leq \exp[-\rho\theta+\sum_{\substack{j=1\\ j\neq i}}^{N} \mathrm{Ln}((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x} C_{ij}(x)dx)] \tag{3.7.2}$$

Let $\{\underline{C}_{ij}(x)\}$ be some element of $\chi(\mathcal{C})$ which minimizes this upper bound to p_f over $\chi(\mathcal{C})$ and let

$$\int_{-\infty}^{\infty} x^2 \underline{C}_{ij}(x)dx = b_j \tag{3.7.3}$$

$\underline{\rho}$ = the optimum ρ relative to $\underline{C}_{ij}(x)$

We have then the lowest upper bound to p_f as

$$\exp[-\underline{\rho}\theta+\sum_{\substack{j=1\\ j\neq i}}^{N} \mathrm{Ln}\ ((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\underline{\rho} x}\underline{C}_{ij}(x)dx)] \tag{3.7.4}$$

Using a power series expansion we can write

$$\int_{-\infty}^{\infty} e^{\rho x} \underline{C}_{ij}(x)dx = \int_{-\infty}^{\infty} (1 + \rho x + \frac{(\rho x)^2}{2!} + \dots) \underline{C}_{ij}(x)dx$$

$$\int_{-\infty}^{\infty} e^{\rho x} \underline{C}_{ij}(x)dx = 1 + \rho \ \underset{\sim}{E}(x) + \frac{\rho^2}{2!} \underset{\sim}{E}(x^2) + \dots \tag{3.7.5}$$

(the expectations are taken with respect to $\underline{C}_{ij}(x)$). Since $\{\underline{C}_{ij}(x)\}$ is an element of $\mathcal{X}(\mathcal{C})$ it must be symmetric. Therefore, all the odd moments of x in (3.7.5) are zero and we have

$$\int_{-\infty}^{\infty} e^{\rho x} \underline{C}_{ij}(x)dx = 1 + \frac{\rho^2}{2} b_j + \sum_{n=2}^{\infty} \frac{\rho^{2n}}{(2n\ !)} \underset{\sim}{E}(x^{2n}) \tag{3.7.6}$$

By Jensen's Inequality we have

$$E(x^{2n}) \geq (E(x^2))^n = b_j^{\ n} \tag{3.7.7}$$

Applying (3.7.7) to (3.7.6) we have

$$\int_{-\infty}^{\infty} e^{\rho x} \underline{C}_{ij}(x)dx \geq 1 + \frac{\rho^2}{2} b_j + \sum_{n=2}^{\infty} \frac{\rho^{2n}}{(2n)!} (\sqrt{b_j})^{2n}$$

$$\int_{-\infty}^{\infty} e^{\rho x} \underline{C}_{ij}(x)dx \geq \int_{-\infty}^{\infty} e^{\rho x} (\tfrac{1}{2} u_0(x - \sqrt{b_j}) + \tfrac{1}{2} u_0(x + \sqrt{b_j}))dx \tag{3.7.8}$$

Now let us define

$$C_{ij}^{+}(x) = \frac{1}{2} u_0(x - \sqrt{b_j}) + \frac{1}{2} u_0(x + \sqrt{b_j})$$

and note that $\{C^{+}_{ij}(x),\ j-1,\ldots N, j\neq i\}$ is an element of $\mathcal{X}(\mathcal{C})$. We have from (3.7.8) that

$$\int_{-\infty}^{\infty} e^{\rho x}\underline{C}_{ij}(x)dx \geq \int_{-\infty}^{\infty} e^{\rho x} C^{+}_{ij}(x)$$

hence,

$$\exp[-\underline{\rho}\theta + \sum_{\substack{j=1\\ j\neq i}}^{N} \mathrm{Ln}((\frac{1-2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x}\underline{C}_{ij}(x)dx)]$$

$$\geq \exp[-\underline{\rho}\theta + \sum_{\substack{j=1\\ j\neq i}}^{N} \mathrm{Ln}((\frac{1-2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x} C^{+}_{ij}(x)dx)] \tag{3.7.9}$$

However, $\{\underline{C}_{ij}(x)\}$ by definition minimizes the right hand side of (3.7.2), thus (3.7.9) must hold with equality and we have

$$\exp[-\underline{\rho}\theta + \sum_{\substack{j=1\\ j\neq i}}^{N} \mathrm{Ln}((\frac{1-2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x}(\frac{1}{2}u_0(x-\sqrt{b_j}) + \frac{1}{2}u_0(x+\sqrt{b_j}))dx)] \tag{3.7.10}$$

as the upper bound to p_f minimized over $\mathcal{X}(\mathcal{C})$ and it is achieved by:

$$C^{+}_{ij}(x) = \frac{1}{2}u_0(x-\sqrt{b_j}) + \frac{1}{2}u_0(x+\sqrt{b_j}) \tag{3.7.11}$$

where

$$\frac{2\tau}{\alpha}\sum_{\substack{j=1\\ j\neq i}}^{N} b_j = p \tag{3.7.12}$$

Of course, we really don't know what $\{\sqrt{b_1},\ldots\sqrt{b_N}\}$ are yet. We now solve for them knowing that they must, by definition, minimize the function

$$\exp[-\rho\theta + \sum_{\substack{j=1\\ j\neq i}}^{N} \mathrm{Ln}((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x}(\frac{1}{2}u_0(x-\sqrt{\beta_j}) + \frac{1}{2}u_0(x+\sqrt{\beta_j}))dx)]$$

over $\{\sqrt{\beta_1},\ldots \sqrt{\beta_j}\}$, $\{\sqrt{\beta_i}$ is excluded$\}$ (3.7.13)

under the constraint

$$\sum_{\substack{j=1\\ j\neq i}}^{N} \beta_j = \frac{p\,\alpha}{2\tau} \tag{3.7.14}$$

Minimizing this function under constraint (3.7.14) is equivalent to minimizing

$$F(\sqrt{\beta_1},\ldots \sqrt{\beta_N}) = \sum_{\substack{j=1\\ j\neq i}}^{N} \mathrm{Ln}((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{\rho x}(\frac{1}{2}u_0(x-\sqrt{\beta_j}) + \frac{1}{2}u_0(x+\sqrt{\beta_j}))dx \tag{3.7.15}$$

under the constraint

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} \beta_j = \frac{p\alpha}{2\tau} \tag{3.7.16}$$

The function given by (3.7.15) is strictly convex (as can be determined by taking the second derivative), in $\{\sqrt{\beta_1},\ldots \sqrt{\beta_N}\}$. Hence, it must have a unique minimum under the constraint (3.7.17). Of course, we know this unique minimum must be

$$\{\sqrt{b_1},\ldots \quad \sqrt{b_N}\} \tag{3.7.17}$$

Now, suppose the components are not all equal. Then a permutation of (3.7.15) will yield another $\{\sqrt{\beta_1},\ldots \quad \sqrt{\beta_N}\}$ which minimizes (3.7.15) under (3.7.16) This contradicts the fact that there is a unique minimum to (3.7.15) Hence, all of the components of (3.7.17) must be equal and since they must satisfy (3.7.16) we have

$$\{\sqrt{b_1},\ldots \quad \sqrt{b_N}\} = \{\sqrt{\frac{p}{(N-1)}\frac{\alpha}{2\tau}},\ldots \sqrt{\frac{p}{(N-1)}\frac{\alpha}{2\tau}}\}$$

Applying this to (3.7.11) we have the upper bound to p_f minimized over $\mathcal{X}(\mathcal{C})$ by $\mathcal{C}^*$

$$\mathcal{C}^* = \{C_{ij}^*(x), j=1,\ldots N, j\neq i\}$$

$$C_{ij}^*(x) = \frac{1}{2}u_0(x-\sqrt{\frac{p\alpha}{(N-1)\,2\tau}}) + \frac{1}{2}u_0(x+\sqrt{\frac{p\alpha}{(N-1)\,2\tau}}) \tag{3.7.18}$$

Observe the lower bound to p_d given by (3.5.13). This can be maximized over $\{C_{ij}(x)\}$ under the average multiple access noise constraint by minimizing

$$\exp[-\gamma(1-\theta) + \sum_{\substack{j=1\\j\neq i}}^{N} \mathrm{Ln}((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} e^{-\gamma x} C_{ij}(x)dx)] \qquad (3.7.19)$$

under this constraint. Going through the same procedure that we used to minimize the upper bound to p_f we find that (3.7.19). is minimized and hence the upper bound is maximized by (3.7.18). One should see this immediately since (3.7.19) has the same form as an upper bound to p_f and $C_{ij}(x)$ is symmetric.

This completes the proof.

If the optimum amplitude density given by (3.7.1) is substituted into the ROC bounds of (3.5.13) one obtains the following ROC upper bound:

$$p_f \leq \exp[-\rho\theta + (N-1)\ \mathrm{Ln}((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\cosh\rho\sqrt{\frac{p\alpha}{(N-1)2\tau}})]$$
$$p_d \geq 1-\exp[-\gamma(1-\theta) + (N-1)\ \mathrm{Ln}((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\cosh\gamma\sqrt{\frac{p\alpha}{(N-1)2\tau}})] \qquad (3.7.20)$$

We can for this optimal case actually solve for the value of ρ which minimizes the upper bound to p_f and also the value of γ which maximizes the lower bound to p_d. If we do this the variation of the optimal ROC with the signal parameters will become explicit. We now devote ourselves to this task.

Let us first define

$$\mathrm{Expf} = [\rho\theta-(N-1)\mathrm{Ln}((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\cosh\rho\sqrt{\frac{p\alpha}{(N-1)2\tau}})] \quad (3.7.21)$$

$$\mathrm{Expd} = [\gamma(1-\theta)-(N-1)\mathrm{Ln}((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\cosh\gamma\sqrt{\frac{p\alpha}{(N-1)2\tau}})] \quad (3.7.22)$$

It is evident that we can express the optimal ROC bounds as

$$p_f \leq \exp(-\mathrm{Expf})$$

$$p_d \geq 1-\exp(-\mathrm{Expd})$$

We need only work with the upper bound to p_f. Our result can be applied to p_d by symmetry.

The ρ which minimizes the upper bound to p_f is the same ρ which maximizes Expf. It is the solution of

$$\frac{\partial}{\partial\rho}\mathrm{Expf} = 0$$

$$\theta - \frac{(N-1)\frac{2\tau}{\alpha}\sqrt{\frac{p\alpha}{(N-1)2\tau}}\sinh\rho\sqrt{\frac{p\alpha}{(N-1)2\tau}}}{(1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\cosh\rho\sqrt{\frac{p\alpha}{(N-1)2\tau}}} = 0$$

Which can be solved directly yielding,

$$\rho = \sqrt{\frac{(N-1)2\tau}{p\alpha}}\ \sinh^{-1}\left[\frac{\theta(1-\frac{2\tau}{\alpha})}{\sqrt{2(N-1)\frac{\tau p}{\alpha} - \frac{\theta^2 4\tau^2}{\alpha^2}}}\right] + \sqrt{\frac{(N-1)2\tau}{p\alpha}}\ \text{Tanh}^{-1}\ \theta\left(\sqrt{\frac{2\tau}{(N-1)p\alpha}}\right)$$

(3.7.23)

Substituting (3.7.23) into (3.7.21) we have

$$\text{Expf} = \theta\sqrt{\frac{(N-1)2\tau}{p\alpha}}\ \sinh^{-1}\left[\frac{\theta(1-\frac{2\tau}{\alpha})}{\sqrt{2(N-1)\frac{\tau p}{\alpha} - \frac{\theta^2 4\tau^2}{\alpha^2}}}\right] + \theta\sqrt{\frac{(N-1)2\tau}{p\alpha}}\ \text{Tanh}^{-1}\left(\theta\sqrt{\frac{2\tau}{(N-1)p\alpha}}\right)$$

$$-(N-1)\text{Ln}\left((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\cosh\left[\sinh^{-1}\left(\frac{\theta(1-\frac{2\tau}{\alpha})}{\sqrt{2(N-1)\frac{\tau p}{\alpha} - \frac{\theta^2 4\tau^2}{\alpha^2}}}\right) + \text{Tanh}^{-1}\left(\theta\sqrt{\frac{2\tau}{(N-1)p\alpha}}\right)\right]\right)$$

(3.7.24)

Going through the same we can obtain for Expd

$$\text{Expd} = (1-\theta)\sqrt{\frac{(N-1)2\tau}{p\alpha}}\ \sinh^{-1}\left[\frac{(1-\theta)(1-\frac{2\tau}{\alpha})}{\sqrt{2(N-1)\frac{\tau p}{\alpha} - \frac{(1-\theta)^2 4\tau^2}{\alpha^2}}}\right] + (1-\theta)\sqrt{\frac{(N-1)2\tau}{p\alpha}}\text{Tanh}^{-1}\ (1-\theta)\sqrt{\frac{2\tau}{(N-1)p\alpha}}$$

$$-(N-1)\text{Ln}\left((1-\frac{2\tau}{\alpha}) + \frac{2\tau}{\alpha}\cosh\left[\sinh^{-1}\left(\frac{(1-\theta)(1-\frac{2\tau}{\alpha})}{\sqrt{2(N-1)\frac{\tau p}{\alpha} - \frac{(1-\theta)^2 4\tau^2}{\alpha^2}}}\right) \quad {}^{-1}\left((1-\theta)\sqrt{\frac{2\tau}{(N-1)p\alpha}}\right)\right]\right)$$

(3.7.25)

The optimum ROC is then given by

$$p_f \leq \exp(-Expf)$$

$$p_d \geq 1-\exp(-Expd)$$

where Expf and Expd are in turn given by (3.7.24) and (3.7.25). These bounds show the explicit dependence of this optimal ROC on the signal parameters τ/α and N.

Typically $\frac{\tau}{\alpha}$ is of the order of 10^{-5}, and N is of the order of 10^5, in which case (3.7.24) and (3.7.25) can be approximated by

$$\begin{aligned} Expf \approx\ & \theta\sqrt{\frac{N2\tau}{p\alpha}} \quad \sinh^{-1}\left[\frac{\theta}{\sqrt{2N\tau p/\alpha}}\right] \\ & - N\ Ln\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\cosh\left[\sinh^{-1}\left(\frac{\theta}{\sqrt{2N\tau p/\alpha}}\right)\right]\right) \\ Expd \approx\ & (1-\theta)\sqrt{\frac{N2\tau}{p\alpha}} \quad \sinh^{-1}\left[\frac{(1-\theta)}{\sqrt{2N\tau p/\alpha}}\right] \\ & -N\left(Ln\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\cosh\left[\sinh^{-1}\left(\frac{(1-\theta)}{\sqrt{2N\tau p/\alpha}}\right)\right]\right)\right) \end{aligned} \qquad (3.7.26)$$

We can now use the following identities

$$\sinh^{-1}(x) = \mathrm{Ln}\left(x + \sqrt{x^2+1}\right)$$

$$\cosh(\sinh^{-1}x) = \sqrt{1+x^2}$$

Applying these to (3.7.26) we obtain

$$\mathrm{Expf} \approx \theta\sqrt{\frac{N2\tau}{p\alpha}}\ \mathrm{Ln}\left(\frac{\theta}{2N\tau p/\alpha} + \sqrt{\frac{\theta^2}{2N\tau p/\alpha} + 1}\right) - N\left(\mathrm{Ln}\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\sqrt{\frac{\theta^2}{2N\tau p/\alpha} + 1}\right)\right) \quad (3.7.27)$$

$$\mathrm{Expd} \approx (1-\theta)\sqrt{\frac{N2\tau}{p\alpha}}\ \mathrm{Ln}\left(\frac{(1-\theta)}{\sqrt{2N\tau p/\alpha}} + \sqrt{\frac{(1-\theta)^2}{2N\tau p/\alpha} + 1}\right)$$

$$- N\left(\mathrm{Ln}\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\sqrt{\frac{(1-\theta)^2}{2N\tau p/\alpha} + 1}\right)\right)$$

which are simpler expressions and they can be applied directly to yield approximate bounds

$$p_f \leq \exp(-\mathrm{Expf})$$

$$p_d \geq 1-\exp(-\mathrm{Expd})$$

3.8 GAUSSIAN COMPARISON

In this section we shall investigate the validity of assuming the multiple access noise to be gaussian distributed. We shall do this under the restriction that the signature set, S, be a uniform signature set (see Definition 3.1) and that the amplitude density is common (i.e. $C_{ij}(x)=C(x)$ for all $j \neq 1$).

Our program in this section will be as follows. We shall first fix N and τ/α and compute the ROC's of several candidate signature sets, (or equivalently common amplitude densities). We will compute the ROC that the matched filter detector would have if its output noise were gaussian having the same average power implied by the fixed values of N and τ/α and the different common amplitude densities. The actual ROC's will then be compared to this gaussian noise ROC. Finally, we shall look at the approach of the ROC corresponding to the Quasi-Optimal amplitude density (of Section 3.7) to the gaussian ROC, with increasing aircraft population, N.

Consider the signature set, S, to be a uniform signature set having a common amplitude density function, C(x), i.e.,

$$C_{ij}(x) = C(x) \quad \text{for all } j \neq i$$

In Theorem 3.1, we state that in this case, if τ/α is fixed, then as N increases the distribution on a multiple access noise sample approaches a zero mean gaussian distribution with variance $\frac{N2\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx$. What is left open though is how rapidly the multiple access noise distribution approaches this gaussian distribution.

If the true multiple access noise distribution is very close to this limit for a typical value of $(N, \tau/\alpha)$, the actual ROC of the matched filter detector operating in the presence of multiple access noise should be closely approximated by ROCg. ROCg is the ROC which the matched filter detector would have if it were operating in an environment in which the output of the matched filter is perturbed only be zero mean gaussian noise having average power $\frac{N2\tau}{\alpha}\int_{-\infty}^{\infty} x^2 C(x)dx$.

Assuming that the signature set, S, is a uniform signature set we shall first test the validity of assuming multiple access noise to be gaussian (when $C_{ij}(x) = C(x)$) by comparing the actual ROC to ROCg for several different $C(x)$'s. We shall fix $(N, \tau/\alpha)$ at a typical value.

First, we shall compute ROCg

<u>P_f calculation for ROCg</u>

$$p_f = \int_{\theta}^{\infty} \frac{1}{\sqrt{2\pi \frac{N2\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx}} \exp\left(\frac{-y^2}{\frac{4N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx} \right) dy$$

$$p_f = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\theta \sqrt{\frac{1}{\frac{4N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx}} \right) \tag{3.8.1}$$

<u>p_d calculation for ROCg</u>

$$p_d = \int_{\theta}^{\infty} \frac{1}{\sqrt{2\pi \frac{N2\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx}} \exp\left(\frac{-(y-1)^2}{\frac{4N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx} \right) dy$$

$$p_d = \frac{1}{2} + \frac{1}{2} \text{erf} \left((1-\theta) \sqrt{\frac{1}{\frac{4N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx}} \right) \tag{3.8.2}$$

Combining (3.8.1) and (3.8.2) we have the ROCg

ROCg

$$p_f = \frac{1}{2} - \frac{1}{2} \text{erf} \left(\theta \sqrt{\frac{1}{\frac{4N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx}} \right)$$

$$p_d = \frac{1}{2} + \frac{1}{2} \text{erf} \left((1-\theta) \sqrt{\frac{1}{\frac{4N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx}} \right) \tag{3.8.3}$$

We shall now compare the ROCg given by (3.8.3) to the bounds to the actual ROC given by (3.5.13) and (3.5.14). In doing this we assume

(i) $N=10^5$

(ii) $\tau/\alpha=2(10^{-5})$

and we consider four different common amplitude densities; $C_1(x),\ldots C_4(x)$. These are shown in Figure 3.2. Bounds to the actual ROC for these amplitude densities are shown in Figures 3.3 through 3.6. The ROCg is shown in Figure 3.8.

Observe the actual ROC's shown in Figures 3.3 through 3.6. In each of these figures the solid contours represent upper bounds to the actual ROC, the broken contours represent lower bound to the actual ROC. For both the actual ROC's of Figures 3.3 through 3.6 and the ROCg of Figure 3.8, each contour corresponds to a given value of $\frac{1}{\sqrt{3b}}$, the value of which is labelled on the contour. Each point on a contour corresponds to a different value of θ. Comparing the corresponding contours on each of the actual ROC sets with that on the ROCg set (i.e. contours having equal values of $1/\sqrt{3b}$, one will note that the ROCg is a much more preferable receiver operating characteristic than the actual ROC. One **concludes** this by fixing $\frac{1}{\sqrt{3b}}$ and p_d and observing the value of p_f indicated by the ROCg and the lower bound to p_f indicated by the lower bound to the actual ROC. The value of p_f indicated by the ROCg is always lower than the actual p_f. Thus, we can conclude that for the values of N and τ/α that we have assumed; (i.e., $N = 10^5$, $\tau/\alpha = 2(10^{-5})$) and for the common densities, $C_1(x),\ldots C_4(x)$; the gaussian approximation to the multiple access noise distribution is a very poor approximation. It indicates much better performance than actually exists.

The common density $C_2(x)$ is the optimal amplitude density derived in the previous section. From this we immediately conclude that if; $N = 10^5$, $\tau/\alpha = 2(10^{-5})$, and the average multiple access noise power is fixed at p **then**

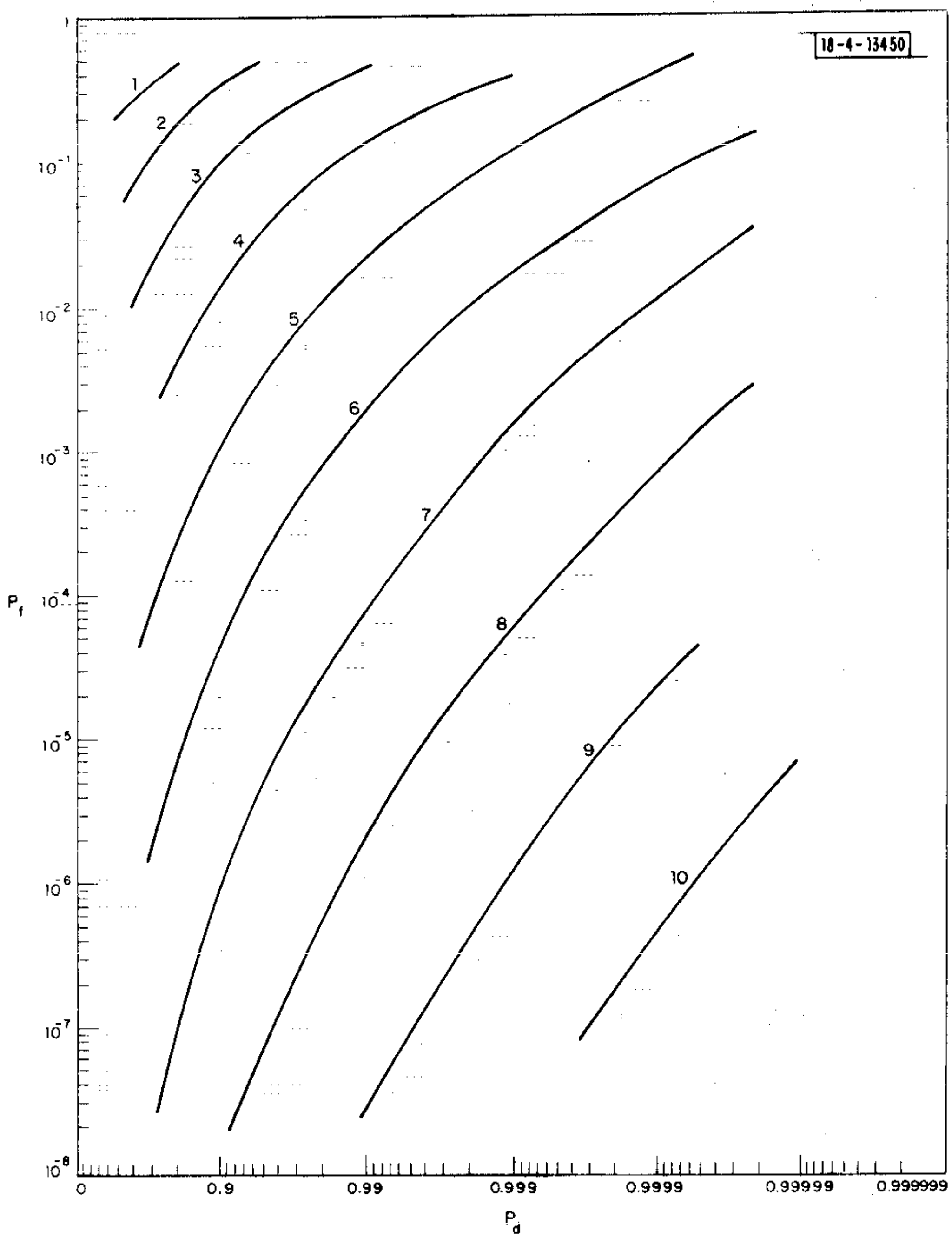

Fig. 3.8. The ROCg for various values of $\frac{1}{\sqrt{3b}}$.

no uniform signature set suffering this average multiple access noise power will yield an ROC more preferable than

$$p_f = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\theta \sqrt{\frac{1}{2p}}\right)$$

$$p_d = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\theta \sqrt{\frac{1}{2p}}\right)$$

In Theorem 3.1 we stated that, if a signature set has a common amplitude density, C(x), then when τ/α is held fixed as N increases the distribution on the multiple access noise approaches a zero mean gaussian distribution of variance $\frac{N2\tau}{\alpha} \int_{\infty}^{-\infty} x^2 \, C(x)dx$. Of course, we have in the present section, showed that for typical parameter values; $N=10^5$, $\tau/\alpha = 2(10^{-5})$, that assuming the multiple access noise distribution to be at the gaussian limit is overly. optimistic. We arrived at this from the observation that ROCg, the ROC under the gaussian assumption, is much more preferable than the actual ROC for these parameter values. When there is a common amplitude density, the actual ROC must approach ROCg with increasing N. In the remainder of this section we shall concern ourselves with studying this approach to ROCg for the case in which the common amplitude density is the Quasi-optimal one, given in Section 3.7, i.e.

$$\mathcal{C}(x) = \frac{1}{2} u_0\left(x - \sqrt{\frac{p\alpha}{(N-1)2\tau}}\right) + \frac{1}{2} u_0\left(x + \sqrt{\frac{p\alpha}{(N-1)2\tau}}\right)$$

In this case, the distribution on the multiple access noise approaches the distribution of a zero mean gaussian random variable of variance p. ROC, of course, is given by

$$p_f = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\theta\sqrt{\frac{1}{2p}}\right)$$

$$p_d = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left((1-\theta)\sqrt{\frac{1}{2p}}\right) \qquad (3.8.4)$$

Because the Chernoff bounds to the actual ROC are asymptotically tight with N, we lose nothing by studying their approach to ROCg. It will be more convenient for us to work with Chernoff bounds to the ROCg rather than the exact ROCg. Again, we lose nothing by doing this since these bounds are asymptotically tight. We shall now derive the Chernoff upper bound to the ROCg. For a given sampling time, t, we have, in general,

$$p_f = \mathrm{Prob}(n_i(t) \geq \theta)$$

Applying a Chernoff bound to this we have

$$p_f \leq \exp(-\mu\theta + \mathrm{Ln}(Ee^{\mu n})) \qquad (3.8.5)$$

where μ is greater than or equal to zero, but is otherwise arbitrary for the present. If the multiple access noise is gaussian

$$E(e^{\mu n}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi p}} \exp\left(-\frac{n^2}{2p}\right)\exp(\mu n)\,dn$$

$$E(e^{\mu n}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi p}} \exp\left(-\frac{(n-\mu p)^2}{2p}\right) \exp\left(\frac{p\mu^2}{2}\right) dn$$

$$E(e^{\mu n}) = \exp\left(\frac{p\mu^2}{2}\right) \qquad (3.8.6)$$

Applying (3.8.6) to (3.8.5) we have

$$p_f \leq \exp -\left(\mu\theta - \frac{p\mu^2}{2}\right)$$

We can now obtain the optimal μ. It is the μ which maximizes $\mu\theta - \frac{p\mu^2}{2}$, solving for it we know that it must satisfy

$$\frac{\partial}{\partial\mu}\left(\mu\theta - \frac{p\mu^2}{2}\right) = 0$$

and hence, we have

$$\mu = \frac{\theta}{p}$$

from which we obtain the Chernoff bound to p_f

$$p_f \leq \exp -\left(\frac{\theta^2}{2p}\right)$$

By going through the same procedure we can obtain

$$p_d \geq 1-\exp-(\frac{(1-\theta)^2}{2p})$$

Hence, the Chernoff upper bound to the ROCg is

$$p_f \leq \exp-(\frac{\theta^2}{2p})$$

$$p_d \geq 1-\exp-(\frac{(1-\theta)^2}{2p}) \tag{3.8.7}$$

Let us define

$$\text{Expfg} = \frac{\theta^2}{2p}$$

$$\text{Expdg} = \frac{(1-\theta)^2}{2p} \tag{3.8.8}$$

We may write the upper bound to the ROCg much more simply as

$$p_f \leq \exp(-\text{Expfg})$$

$$p_d \geq 1-\exp(-\text{Expdg}) \tag{3.8.9}$$

In Section 3.7, we show that the upper bound to the actual ROC under the optimal amplitude density $\mathcal{C}(x)$:

$$\mathcal{C}(x) = \frac{1}{2}u_0\left(x-\sqrt{\frac{p\alpha}{(N-1)2\tau}}\right) + \frac{1}{2}u_0\left(x+\sqrt{\frac{p\alpha}{(N-1)2\tau}}\right)$$

was given by

$$p_f \leq \exp(-\mathrm{Expf})$$

$$p_d \geq 1-\exp(-\mathrm{Expd}) \qquad (3.8.10)$$

Expf and Expd are given approximately by (3.7.27).

We can study the approach of, the upper bound to the actual ROC, to the upper bound to ROCg, by studying the approach of Expf to Expfg and Expd to Expdg. We shall now do this. It will be convenient to express; Expf in terms of Expfg, and Expd in terms of Expdg. This is quite easy to do and we obtain

$$\mathrm{Expf} = \sqrt{\frac{4N\tau}{\alpha}\mathrm{Expfg}}\ \mathrm{Ln}\left(\sqrt{\frac{\mathrm{Expfg}}{\frac{N\tau}{\alpha}}} + \sqrt{1+\frac{\mathrm{Expfg}}{\frac{N\tau}{\alpha}}}\right)$$
$$-N\ \mathrm{Ln}\left[\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\sqrt{1+\frac{\mathrm{Expfg}}{\frac{N\tau}{\alpha}}}\right]$$

$$\mathrm{Expd} = \sqrt{\frac{4N\tau}{\alpha}\mathrm{Expdg}}\ \mathrm{Ln}\left(\sqrt{\frac{\mathrm{Expdg}}{\frac{N\tau}{\alpha}}} + \sqrt{1+\frac{\mathrm{Expdg}}{\frac{N\tau}{\alpha}}}\right)$$
$$-N\ \mathrm{Ln}\left[\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\sqrt{1+\frac{\mathrm{Expdg}}{\frac{N\tau}{\alpha}}}\right] \qquad (3.8.11)$$

Assume that τ/α, Expfg and Expdg are fixed. If N is increased then we expect to see Expf approach Expfg and Expd approach Expdg. Let us check to see if this is true. Using the Taylor expansion for $\sqrt{1+x}$ for N is sufficiently large to enable $\frac{Expfg}{N\tau/\alpha}$ and $\frac{Expdg}{N\tau/\alpha}$ to be within the radius of convergence we obtain

$$\sqrt{1+\frac{Expfg}{N\tau/\alpha}} = 1 + \frac{1}{2}\left(\frac{Expfg}{N\tau/\alpha}\right) - \frac{1}{8}\left(\frac{Expfg}{N\tau/\alpha}\right)^2 + \frac{1}{16}\left(\frac{Expfg}{N\tau/\alpha}\right)^3 + \ldots$$

$$\sqrt{1+\frac{Expdg}{N\tau/\alpha}} = 1 + \frac{1}{2}\left(\frac{Expdg}{N\tau/\alpha}\right) - \frac{1}{8}\left(\frac{Expdg}{N\tau/\alpha}\right)^2 + \frac{1}{16}\left(\frac{Expdg}{N\tau/\alpha}\right)^3 + \ldots \qquad (3.8.12)$$

and from this we note that

$$\sqrt{\frac{Expfg}{N\tau/\alpha}} + \sqrt{1+\frac{Expfg}{N\tau/\alpha}} \sim 1 + \sqrt{\frac{Expfg}{N\tau/\alpha}} \qquad \text{as } N\to\infty$$

$$\sqrt{\frac{Expdg}{N\tau/\alpha}} + \sqrt{1+\frac{Expdg}{N\tau/\alpha}} \sim 1 + \sqrt{\frac{Expdg}{N\tau/\alpha}} \qquad \text{as } N\to\infty$$

$$\sqrt{1+\frac{Expfg}{N\tau/\alpha}} \sim 1 + \frac{1}{2}\left(\frac{Expfg}{N\tau/\alpha}\right) \qquad \text{as } N\to\infty$$

$$\sqrt{1+\frac{Expdg}{N\tau/\alpha}} \sim 1 + \frac{1}{2}\left(\frac{Expdg}{N\tau/\alpha}\right) \qquad \text{as } N\to\infty$$

Applying these asymptotic relations to (3.8.11) we obtain

$$\text{Expf} \sim \sqrt{\frac{4N\tau}{\alpha}\text{Expfg}}\ \text{Ln}\left(1 + \sqrt{\frac{\text{Expfg}}{N\tau/\alpha}}\right) \qquad \text{as } N\to\infty$$

$$- N\ \text{Ln}\left(1 + \frac{\tau}{\alpha}\left(\frac{\text{Expfg}}{N\tau/\alpha}\right)\right) \tag{3.8.13}$$

$$\text{Expd} \sim \sqrt{\frac{4N\tau}{\alpha}\text{Expdg}}\ \text{Ln}\left(1 + \sqrt{\frac{\text{Expdg}}{N\tau/\alpha}}\right) \qquad \text{as } N\to\infty$$

$$- N\ \text{Ln}\left(1 + \frac{\tau}{\alpha}\left(\frac{\text{Expdg}}{N\tau/\alpha}\right)\right) \tag{3.8.14}$$

Since

$$\text{Ln}(1 + x) \sim x$$

for x near zero

We obtain from (3.8.13) and (3.8.14)

$$\text{Expf} \sim \text{Expfg} \qquad \text{as } N\to\infty \tag{3.8.15}$$

$$\text{Expd} \sim \text{Expdg} \qquad \text{as } N\to\infty$$

This, of course, is what we expected from Theorem 3.1.

Observe the expressions given by (3.8.11) notice that they are of the same form. For this reason we shall represent Expf and Expd by Exp, Expfg and Expdg by Expg and use the following expression to represent (3.8.11).

$$\text{Exp} = \sqrt{\frac{4N\tau}{\alpha}\text{Expg}}\ \text{Ln}\left(\sqrt{\frac{\text{Expg}}{\frac{N\tau}{\alpha}}} + \sqrt{1 + \frac{\text{Expg}}{\frac{N\tau}{\alpha}}}\right)$$

$$- N\ \text{Ln}\left[\left(1 - \frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\sqrt{1 + \frac{\text{Expg}}{\frac{N\tau}{\alpha}}}\ \right] \qquad (3.8.16)$$

Figure 3.9 illustrates Exp as a function of N for Expg = 4 and various values of τ/α. Notice that in each case, Exp increases monotonically and very smoothly with N up to the gaussian exponent Expg. Also, notice that when, $N=10^5$, $\tau/\alpha = 10^{-5}$, typical system parameter values, Exp is less than Expg by a non-negligible amount.

In Figure 3.10 we have plotted Exp vs. Expg for $\tau/\alpha = 10^5$ and various values of N. Notice that for any value of Expg, Exp approaches Expg monotonically from below with increasing N. Notice also that the rate of this approach is more rapid for lower values of Expg than for higher values of Expg. Finally, observe that the functional dependence between Exp and Expg is practically linear.

The smoothness and the monotonicity properties of the curves shown in Figures 3.9 and 3.10 seem to indicate that a simple relation, (simpler than (3.8.16)), exists between Exp and Expg and that Exp is always less than or equal to Expg.

We can use the curve shown in Figure 3.10 to judge the validity of the way in which the multiple access noise degradation was accounted

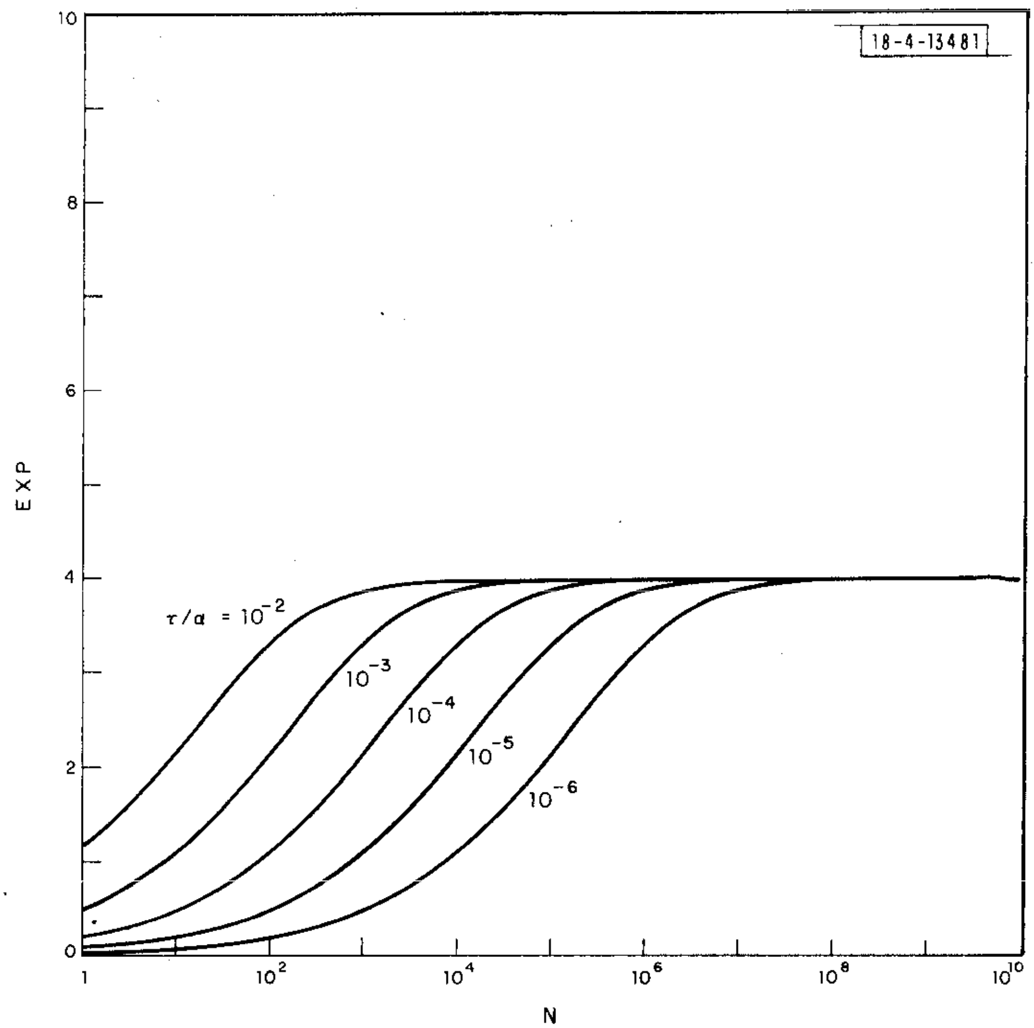

Fig. 3.9. Exp vs N when Expg = 4, for various values of τ/α.

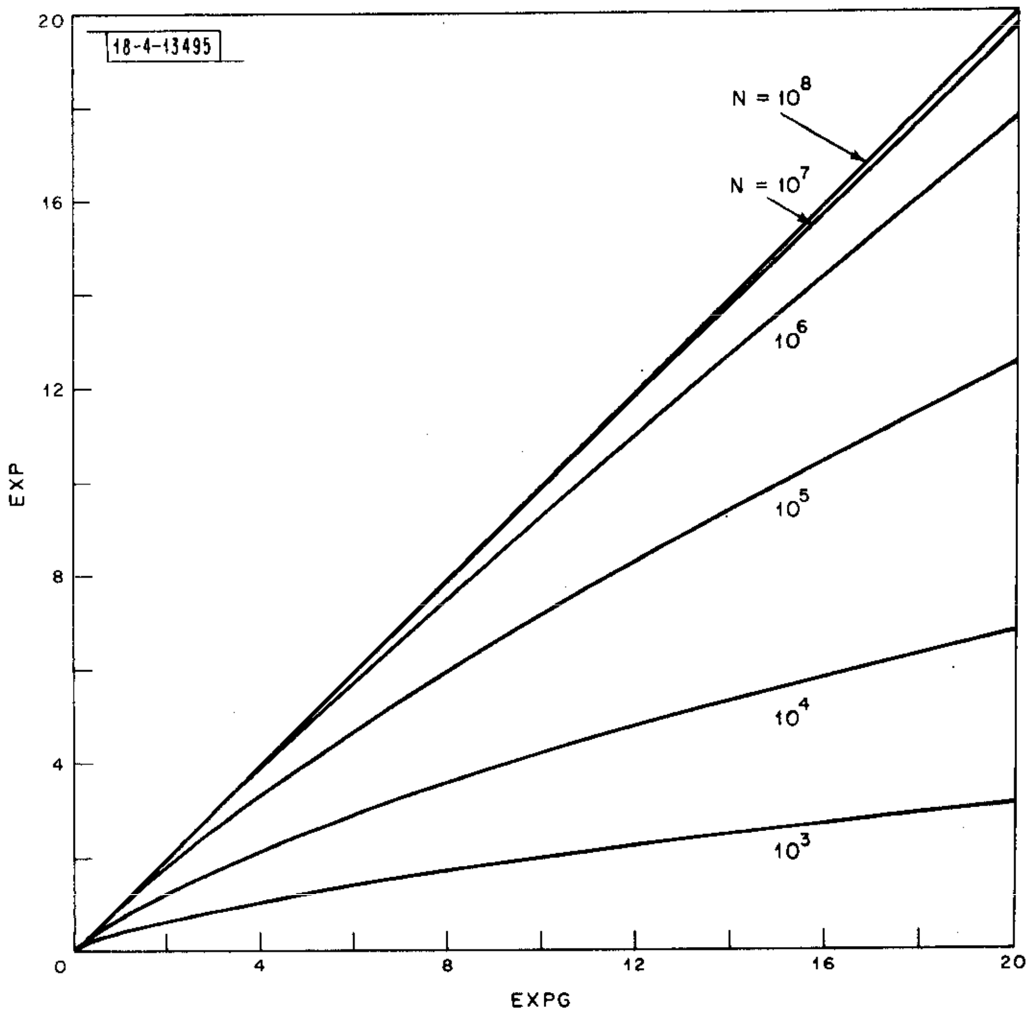

Fig. 3.10. Exp vs Expg when $\tau/\alpha = 10^{-5}$, for various values of N.

for in Reference [1] where it was assumed to have the same effect as an equal power in band white gaussian noise source at the input to each matched filter. Let B represent signal bandwidth. The average multiple access noise power under such an assumption would be

$$p = \frac{N-1}{B\alpha}$$

This assumes, without loss of generality, unit energy signals. In the report,[1] systems were considered in which $N = 10^5$. However, signatures were assumed to consist of either 5 equal length pulses with α=2.5 seconds or 4 equal length pulses with α= 2 seconds. This would make N effectively equal to $2(10^5)$. B was assumed to be 10^7Hz. All of this yields

$$p = 50$$

Now,

$$\text{Expg} = \frac{a^2}{P}$$

where $a=1-\theta$ or θ depending upon whether we are dealing with p_f or p_d. Suppose, as an example, we let $\theta = 0.5$ then

$$\text{Expg} = 12.5$$

In the system considered in Reference [1], τ was always of the order of 10 μsec. If we assume that these signature sets had correlation properties which made them uniform signature sets, the Exp would correspond to the best p_f or p_d exponent which one could expect. Observing Figure 3.10 which corresponds to a $\tau/\alpha = 10^{-5}$. If we let $N=2\ (10^{-5})$ and Expg=12.5, we see that Exp$\approx$9. This implies that the gaussian assumption underlying the analysis in [1] yielded a much more optimistic system performance than actually existed when one considers that in this situation the ratio of Expg to Exp is approximately 1.4.

3.9 RELATION OF MULTIPLE ACCESS NOISE TO TIME-BANDWIDTH PRODUCT

This section will deal with the relation of "Time-Bandwidth" product to the average multiple access noise power from filter "i". In doing these things we shall be able to find limits on the performance of the matched filter detector, operating in the presence of multiple access noise, in terms of a traditional measure of such performance. We shall close this section by applying the results obtained to a signature set, S, composed of PSK modulated sequences having good auto-correlation properties.

The following theorem relating the average multiple access noise power is based upon a result of Yates[4].

Theorem 3.3: Let S be a signature set and let $E(n^2)$ be the average multiple access noise power corresponding to it. The following upper bound holds

$$E(n^2) \leq \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{1}{4\alpha \sqrt{W_i W_j}}$$

where W_j is the Zakai bandwidth of signal $S_j(t)$ and is defined as

$$W_j = \left[2 \int_{-\infty}^{\infty} R_{jj}^2 (t)\, dt\right]^{-1}$$

and $R_{jj}(t)$ is the auto-correlation of signal $S_j(\cdot)$, i.e.,

$$R_{jj}(t) = \int_{-\infty}^{\infty} S_j(x)\, S_j(x - t)\, dx \qquad (3.9.1)$$

The bound holds with equality iff $R_{jj}(\cdot) = R_{ii}(\cdot)$ for all $j \neq i$.

The proof of this theorem is given in Appendix F.

Consider now a signature set, S^+, which is composed of PSK modulated sequences. This was the type of signature set which was considered in Reference [1]. Furthermore, suppose that the chip duration of these PSK sequences is τ_c and that these sequences all have the same auto-correlation function, R(t), which is illustrated in Figure 3.11. The auto-correlation was defined by (3.9.1). The common auto-correlation shown in Figure 3.9.1 is a "good auto-correlation function" in the sense that it has a high peak to side lobe ratio. This also was implicit in the analyses of Reference [1].

Let us apply Theorem 3.3 to signature set S^+. Because of the common auto-correlation, the theorem applys with equality and we have for the average multiple access noise power

$$E(n^2) = \frac{(N-1)}{4\alpha W} \tag{3.9.2}$$

where W is the common Zakai bandwidth. Computing the common Zakai Bandwidth we obtain

$$W = \left[\frac{4}{3} \tau_c \left(1 + 2 \frac{\tau_c}{\tau} + 2 \left(\frac{\tau_c}{\tau}\right)^2\right]^{-1}$$

If we consider $\frac{\tau_c}{\tau} \ll 1$ then

$$W \simeq \frac{3/4}{\tau_c} \tag{3.9.3}$$

Applying (3.9.3) to (3.9.2) yields

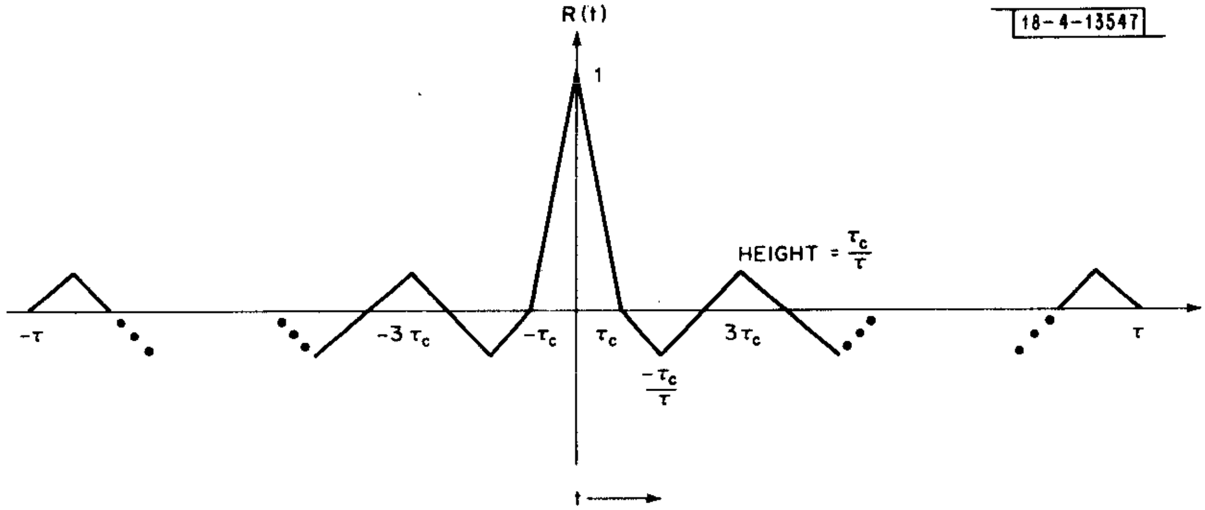

Fig. 3. 11. Good auto-correlation functions for a PSK sequence.

$$E(n^2) \approx \frac{N}{3\alpha \; 1/\tau_c}$$

This is the same average multiple access noise power which would be obtained if the interference input to the matched filter detector was white gaussian noise having a spectral height of $\frac{N}{3\alpha \; 1/\tau_c}$. This of course is very close to what Stiglitz et. al. assumed in [1]. The peak signal power of signature set S^+ is unity. Applying this to (3.9.2) yields

$$\frac{3\alpha \; 1/\tau_c}{N}$$

as the peak signal to interference ratio.

SECTION 4

IMPROVED AIRCRAFT POSITION TRACKING WITH SATELLITE MULTILATERATION SURVEILLANCE SYSTEMS

4.1 INTRODUCTION

Several proposed air traffic control systems would make use of satellite multilateration techniques to satisfy the surveillance requirements. In these systems, each aircraft, while in flight, would transmit a sequence of coded pulses at the more or less regularly spaced time instants τ_0, τ_1, τ_2, ..., where $\tau_{i+1}-\tau_i \cong 1$ second. A constellation of K satellites in synchronous orbits would receive each pulse; and, by comparing the times of arrival of this pulse at the K satellites, an estimate of the position of the aircraft at the time the pulse was transmitted could then be computed. Thus, the surveillance system would estimate the sequence $p(\tau_0)$, $p(\tau_1)$, $p(\tau_2)$, ..., where the 3-vector $p(t)$ denotes the aircraft position at time t.

An analysis of the geometric dilution error has been presented in Appendix I of [1]. For that analysis, no attempt was made to statistically model either the sequence $\{\tau_i\}$ or the aircraft trajectory $p(t)$ or to exploit such a model with a tracking algorithm. It was shown that if the time of arrival of a pulse at each of the K satellites were measured with resulting error variances equal to σ^2, then the covariance matrix for ξ_p, the error in estimating the aircraft position p, would just be

$$P_{\xi_p} = \sigma^2[F'H'(H\ H')^{-1}H\ F]^{-1} \tag{4.1.1}$$

In the above,H is an arbitary K-1 by K matrix subject to the constraints that its rows are indendendent and its columns sum to zero; while F is a K by 3 matrix for which the i^{th} row is $\frac{1}{c}\ u_i'$, where u_i is the unit vector pointing from satellite i to the aircraft, and c is the speed of light.

From (4.1.1) it follows that the variance of the norm of the error ξ_p is just

$$\begin{aligned} Var[||\xi_p||] &= \sigma^2 tr\{[F'H'(HH')^{-1}HF]^{-1}\} \\ &= \sigma^2 k^2 \end{aligned} \tag{4.1.2}$$

where tr $\{\cdot\}$ denotes the trace operation. The constant k, which is the ratio of rms position estimation errors to rms errors in measuring the pulse times of arrival at the satellites, is called the "geometric dilution" of the surveillance system. It should be noted that k is finite if and only if rank (HF)=3; a necessary condition for this to be true is that $K \geq 4$. However, it was shown in Appendix D of [1] that typical satellite constellations of from five to eight satellites could be expected to have geometric dilutions of from one to two orders of magnitude.

It is reasonable to expect that if suitable stochastic models for the sequence $\{\tau_i\}$ and the aircraft trajectory p(t) can be found, then the estimation technique of [1], Appendix I could be improved. This, in fact, is the case as we shall now demonstrate.

It will be shown in the remainder of this section that by incorporating a suitable model for the sequence $\{\tau_i\}$, the variance of the error ξ_p can be reduced. Moreover, it will now also be possible to track an aircraft, over sufficiently short intervals of time, using only three satellites.

4.2 THE MULTILATERATION EQUATIONS

It is assumed that the aircraft is to be tracked using a constellation of K statellites. Further, we shall use the following notation:

$\{\tau_n\}$ = sequence of aircraft pulse transmission times

$p(t)$ = aircraft position vector at time t

p_n = $p(\tau_n)$

T_{ni} = time of arrival of pulse n at satellite i

T_n = $(T_{n1}, \ldots T_{nK})'$

$S_i(t)$ = position of satellite i at time t.

Also, it will be convenient to denote the K-vector, all of whose components are 1's, simply as 1.

With the above definitions, the fundamental multilateration equation is

$$T_n = \tau_n 1 + \frac{1}{c}\begin{pmatrix} ||p_n - S_1(T_{n1})|| \\ \vdots \\ ||p_n - S_K(T_{nK})|| \end{pmatrix} + \eta_n \tag{4.2.1}$$

where η_n is a K-vector which accounts for excess delays due to ionospheric and tropospheric effects. We shall decompose η_n into two components, a deterministic, (i.e., known) part and a random part:

$$\eta_n = \eta_{dn} + \eta_{rn}$$

Of course, the times of arrival of the n^{th} pulse at the satellites must be measured. Thus, let $\hat{T}_n$ denote the measured value of the vector T_n, and ξ_{T_n}, the resulting error, so that

$$\xi_{T_n} = \hat{T}_n - T_n$$

With this definition, (4.2.1) becomes

$$\hat{T}_n = \tau_n 1 + \frac{1}{c}\begin{pmatrix} ||p_n - S_1(T_{n1})|| \\ \vdots \\ ||p_n - S_K(T_{n1})|| \end{pmatrix} + \eta_{dn} + \eta_{rn} + \xi_{T_n} \tag{4.2.2}$$

Equation (4.2.2) is nonlinear in p_n. Moreover, since neither the times T_{ni} nor the satellite trajectories $S_i(t)$ are known precisely, we must approximate each $S_i(T_{ni})$ by $\hat{S}_i(\hat{T}_{ni})$, where $\hat{S}_i(t)$ denotes the estimated satellite trajectory. Therefore, letting p_n^* denote a good guess of the actual value of p_n(e.g., p_n^* could be the estimated value of $p(\tau_{n-1})$), we can write

$$\|p_n - S_i(T_{ni})\| = \|p_n^* - \hat{S}_i(\hat{T}_{ni}) + p_n - p_n^* + \hat{S}_i(\hat{T}_{ni}) - S_i(T_{ni})\|$$

$$\cong \|p_n^* - \hat{S}_i(\hat{T}_{ni})\| + u_{ni}'(p_n - p_n^*) + u_{ni}'(\hat{S}_i(\hat{T}_{ni}) - S_i(T_{ni}))$$

where u_{ni} is the unit vector pointing from $\hat{S}_i(\hat{T}_{ni})$ to p_n^*:

$$u_{ni} = \frac{1}{\|p_n^* - \hat{S}_i(\hat{T}_{ni})\|} (p_n^* - \hat{S}_i(\hat{T}_{ni}))$$

With this approximation (4.2.2) becomes

$$\hat{T}_n = \tau_n 1 + \frac{1}{c}\begin{pmatrix} \|p_n^* - \hat{S}_1(\hat{T}_{n1})\| \\ \vdots \\ \|p_n^* - \hat{S}_K(\hat{T}_{nK})\| \end{pmatrix} + F_n(p_n - p_n^*)$$

$$+ \frac{1}{c}\begin{pmatrix} u_{n1}'(\hat{S}_1(\hat{T}_{n1}) - S_1(T_{n1})) \\ \vdots \\ u_{nK}'(\hat{S}_K(\hat{T}_{nK}) - S_K(T_{nK})) \end{pmatrix} + \eta_{dn} + \eta_{rn} + \xi_{T_n}$$

where F_n is the following K by 3 matrix:

$$F_n = \frac{1}{c}\begin{pmatrix} u'_{n1} \\ u'_{n2} \\ \vdots \\ u'_{nK} \end{pmatrix}$$

This may be simplified as

$$\hat{T}_n = k_n + \tau_n 1 + F_n (p_n - p_n^*) + \varepsilon_n$$

where k_n is the following deterministic vector:

$$k_n = \frac{1}{c}\begin{pmatrix} \|p_n^* - \hat{S}_1(\hat{T}_{n1})\| \\ \vdots \\ \|p_n^* - \hat{S}_K(\hat{T}_{nK})\| \end{pmatrix} + \eta_{dn}$$

and ε_n is the following random vector:

$$\varepsilon_n = \frac{1}{c}\begin{pmatrix} u'_{n1}(\hat{S}_1(\hat{T}_{n1}) - S_1(T_{n1})) \\ \vdots \\ u'_{nK}(\hat{S}_K(\hat{T}_{nK}) - S_K(T_{nK})) \end{pmatrix} + \eta_{rn} + \xi_{Tn}$$

The sequence of random vectors $\{\varepsilon_n\}$ represents the combined effects of random refractions, satellite tracking errors, and errors in measuring the pulse times of arrival. However, it can be argued that the statistics of $\{\varepsilon_n\}$ may be adequately modeled as

$$E[\varepsilon_n] = 0$$

$$E[\varepsilon_n \varepsilon_m'] = \sigma^2 I \delta_{n,m}$$

where

$$\begin{aligned}\sigma^2 = &[\text{Variance in measuring } T_{ni}] \\ &+ [\text{Variance of } i^{th} \text{ component of } \eta_{rn}] \\ &+ \frac{1}{c^2} \cdot [\text{Variance of error in the radial direction in tracking the } i^{th} \text{ satellite}]\end{aligned}$$

We have, therefore, reduced the multilateration equations to the linear equation

$$\hat{T}_n = k_n + \tau_n 1 + F_n(p_n - p_n^*) + \varepsilon_n \tag{4.2.3}$$

where k_n is a known vector, and $\{\varepsilon_n\}$ is modeled as

$$E[\varepsilon_n] = 0$$

$$E[\varepsilon_n \varepsilon_m'] = \sigma^2 I \delta_{n,m} \tag{4.2.4}$$

All that remains is to model the sequences $\{\tau_n\}$ and $\{p_n\}$; given models for these sequences, we can determine how to estimate them.

4.3 THE MODEL FOR $\{\tau_n\}$

It is shown in Appendix **G** that the random sequence $\{\tau_n\}$ may be modeled quite realistically as having the following second order statistics:

$$E[\tau_n - \tau_o] = n\,\bar{\tau}$$

$$E[(\tau_n - \tau_o)(\tau_m - \tau_o)] = \sigma_1^2 \min(m,n) + \frac{1}{2}\sigma_2^2 (m^{2\rho} + n^{2\rho} - |m-n|^{2\rho}) + \bar{\tau}^2\, m\, n$$

where

$$\sigma_1^2 \geq \sigma_2^2$$

$$\frac{1}{2} < \rho < 1 \ , \ \rho \cong 1$$

However, in order to simplify some of the subsequent analysis, we shall assume that

$$\sigma_1^2 = \sigma_2^2 \triangleq \sigma_0^2$$

and that

$$\rho = 1$$

With these assumptions, $\{\tau_n\}$ is modelled as

$$\tau_o = \text{completely unknown}$$

$$E[\tau_n - \tau_o] = n\bar{\tau} \tag{4.3.1a}$$

$$E[(\tau_m - \tau_o)(\tau_m - \tau_o)] = \sigma_o^2[\min(m,n) + m\ n] + \bar{\tau}^2\ m\ n \tag{4.3.1b}$$

$$\text{Var}\ [\tau_n - \tau_o] = \sigma_o^2(n+n^2) \tag{4.3.1c}$$

Note that, as n increases, the standard deviation of $(\tau_n - \tau_o)$ becomes approximately $n\sigma_o$. Thus, the rms fractional error in $\{\tau_n\}$ is just $\sigma_o/\bar{\tau}$.

4.4 THE MODEL FOR $\{p_n\}$

Since a realistic stochastic model for the aircraft trajectory would be quite complex, the sequence $\{p_n\}$ will be modelled simply as a sequence of completely unknown, i.e., nonrandom, vectors. This assumption considerably simplifies the problem of estimating the vectors p_n. Of course, due to physical limitations on the velocity and acceleration of an aircraft, knowing the value of p_n tells us something about the value of p_{n+1}. Such considerations can be useful in calculating the vectors p_n^*, and in determining the confidence that we may place on a particular estimate $\hat{p}_n$.

It may be noted that this model is the same as that used in [1]. Thus, any improvement in tracking the sequence $\{p_n\}$ will be due to the incorporation of a stochastic model for the sequence $\{\tau_n\}$.

4.5 ESTIMATING THE SEQUENCES $\{\tau_n\}$ AND $\{p_n\}$

At approximately every $\bar{\tau} \cong 1$ second, a set of measurements of pulse times of arrival denoted by (4.2.3) is made. From these measurements we would like to track the sequences $\{\tau_n\}$ and $\{p_n\}$. There are now three basic situations under which we shall use different estimation procedures:

(i) $n = 0$, $K \geq 4$

(ii) $n \geq 1$, $K \geq 4$

(iii) $n \geq N + 1$, $K = 3$, where up to time N there are 4 or more satellites available, but after time N there are only 3.

4.5.1 Initial Estimates of τ_0 and p_0 with Four or More Satellites

We have the measurements

$$\hat{T}_0 = k_0 + \tau_0 1 + F_0(p_0 - p_0^*) + \varepsilon_0$$

and, since both τ_0 and p_0 are completely unknown, a reasonable estimate to use is a least-squares estimate. That is, we pick $\hat{\tau}_0$ and $\hat{p}_0$ so as to minimize

$$(\hat{T}_0 - k_0 - \hat{\tau}_0 1 - F_0(\hat{p}_0 - p_0^*))' \, P_{\varepsilon_0}^{-1} \, (\hat{T}_0 - k_0 - \hat{\tau}_0 1 - F_0(\hat{p}_0 - P_0^*))$$

where P_{ε_0} is the covariance matrix for ε_0. But, from (4.2.4)

$$P_{\varepsilon_0} = \sigma^2 I$$

so the estimates $\hat{\tau}_0$ and $\hat{p}_0$ are given by

$$\begin{pmatrix} \hat{\tau}_0 \\ \hat{p}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ p_0^* \end{pmatrix} + [(1:F_0)'(1:F_0)]^{-1}(1:F_0)'(\hat{T}_0 - k_0)$$

Then, since

$$1'1 = K = \text{number of satellites},$$

the above is just

$$\begin{pmatrix} \hat{\tau}_o \\ \hat{p}_o \end{pmatrix} = \begin{pmatrix} 0 \\ p_o^* \end{pmatrix} + \begin{pmatrix} K & 1'F_o \\ F_o'1 & F_o'F_o \end{pmatrix}^{-1} \begin{pmatrix} 1' \\ F_o' \end{pmatrix} (\hat{T}_o - k_o)$$

It is easily verified that

$$\begin{pmatrix} K & 1'F_o \\ F_o'1 & F_o'F_o \end{pmatrix}^{-1} = \begin{pmatrix} \gamma_o & -\gamma_o 1'F_o(F_o'F_o)^{-1} \\ -\gamma_o(F_o'F_o)^{-1}F_o'1 & (F_o'F_o)^{-1} + \gamma_o(F_o'F_o)^{-1}F_o'11'F_o(F_o'F_o)^{-1} \end{pmatrix} \quad (4.5.1)$$

where

$$\gamma_o = \frac{1}{K-1'F_o(F_o'F_o)^{-1}F_o'1} \quad (4.5.2)$$

Therefore, the estimate $\hat{\tau}_o$ is just

$$\hat{\tau}_o = \gamma_o 1'(I-F_o(F_o'F_o)^{-1}F_o')\ (\hat{T}_o-k_o) \quad (4.5.3)$$

and p_o is

$$\begin{aligned} \hat{p}_o &= p_o^* + (F_o'F_o)^{-1}F_o'[I-\gamma_o 11'(I-F_o(F_o'F_o)^{-1}F_o')]\ (\hat{T}_o-k_o) \\ &= p_o^* + (F_o'F_o)^{-1}F_o'[\hat{T}_o-k_o-\hat{\tau}_o 1] \end{aligned} \quad (4.5.4)$$

It is easily seen that

$$\begin{pmatrix} \hat{\tau}_o \\ \hat{p}_o \end{pmatrix} - \begin{pmatrix} \tau_o \\ p_o \end{pmatrix} = [(1:F_o)'(1:F_o)]^{-1}(1:F_o)' \quad \varepsilon_o$$

Therefore, the error covariance matrix for this combined estimate is just

$$\sigma^2[(1:F_o)' \ (1:F_o)]^{-1}$$

Denoting the variance in the estimate $\hat{\tau}_o$ by $\sigma^2_{\xi_{\tau_o}}$, and the covariance matrix for the estimate $\hat{p}_o$ by $P_{\xi_{p_o}}$ it now follows from (4.5.1) that

$$\sigma^2_{\xi_{\tau_o}} = \frac{\sigma^2}{K-1'F_o(F_o'F_o)^{-1}F_o'1} = \sigma^2 \gamma_o \tag{4.5.5}$$

and that

$$P_{\xi_{p_o}} = \sigma^2(F_o'F_o)^{-1} + \sigma^2_{\xi_{\tau_o}}(F_o'F_o)^{-1}F_o'11'F_o(F_o'F_o)^{-1} \tag{4.5.6}$$

From the latter it follows that

$$\mathrm{Var}[\|\xi_{p_o}\|] = \sigma^2 \ \mathrm{tr}\{(F_o'F_o)^{-1}\} + \sigma^2_{\xi_{\tau_o}} 1'F_o(F_o'F_o)^{-2}F_o'1 \tag{4.5.7}$$

As one might expect, (4.5.6) and (4.1.1) are equivalent expressions for $P_{\xi_{p_o}}$, and (4.5.7) and (4.1.2) are equivalent expressions for $\mathrm{Var}[\|\xi_{p_o}\|]$. In fact, it is easy to show that $\sigma^2_{\xi_{\tau_o}}$ is finite if and only if the geometric dilution, as defined in Section 4.1, is finite.

4.5.2 Tracking $\{\tau_n\}$ and $\{p_n\}$ for $n \geq 1$ with Four or More Satellites

In this case, we have measurements of the form

$$\hat{T}_n = k_n + \tau_n 1 + F_n(p_n - p_n^*) + \varepsilon_n \tag{4.5.8}$$

and, in addition, we have the previously computed estimates $\hat{\tau}_0, \hat{\tau}_1, \ldots \hat{\tau}_{n-1}$. Knowledge of these estimates will allow us to estimate τ_n with a smaller error variance than that given by (4.5.5). This estimate $\hat{\tau}_n$ will, in turn, allow us to estimate p_n with a smaller error covariance matrix than that given by (4.5.6).

The problem of tracking $\{\tau_n\}$ could be approached from the Kalman filter point of view. In fact, it is easy to show that the sequence $\{\tau_n\}$, generated as

$$\left.\begin{aligned} \tau_{n+1} &= 2\tau_n + \alpha_n + w_{n+1} \\ \alpha_{n+1} &= -\tau_n - w_{n+1} \end{aligned}\right\} \quad n \geq 0 \tag{4.5.9}$$

will have the second order statistics of (4.3.1) if

$$E[w_n] = 0; E[w_n w_m] = \sigma_0^2 \, \delta_{n,m}$$

$$E[\tau_0] = 0; E[\tau_0^2] = 0$$

$$E[\alpha_0] = 0; E[\alpha_0^2] = \sigma_0^2$$

It would then be a simple exercise to find the Kalman filter to track the sequence $\{\tau_n\}$.

This approach will not be taken, for the following important reason. Equations (4.3.1) represent the second order statistics of $\{\tau_n\}$ only for the limiting case when $\rho=1$ (see subsection 4.3 and Appendix G). In this case, the sequence $\{\tau_n\}$ is, from (4.5.9), just

$$\tau_n = \sum_{i=1}^{n} w_i + \alpha_0$$

A Kalman filter based on (4.5.9) will tend to estimate both τ_n and α_0, with the estimates for τ_n depending on those for α_0. Because α_0 is a random variable, the estimates for α_0 will converge after a short time, and from that time on the estimates for τ_n will depend on this limiting value.

In the actual case when ρ is strictly less than one, the dynamics of (4.5.9) are <u>not</u> a good model for $\{\tau_n\}$. Therefore, the random variable α_0 has no relevance to the process $\{\tau_n\}$; and therefore, basing estimates for τ_n on an estimate of α_0 is clearly absurd. In short, if one were to use a Kalman filter to track $\{\tau_n\}$, the following phenomenon would be observed: initially, while "α_0" is being estimated, good estimates of τ_n would result; however, after the estimate for "α_0" has converged, poor estimates of τ_n would result.

To circumvent this problem, while at the same time using the simple second order statistics of (4.3.1), we shall take the following approach. We shall constrain the estimate $\hat{\tau}_n$ to be a function only of the current measurement, $\hat{T}_n$, and the most recent past estimate, $\hat{\tau}_{n-1}$. Further, we shall constrain this function to be linear.

We begin by estimating τ_1. Define the increment Δ_1 as

$$\Delta_1 = \tau_1 - \tau_0 - \overline{\tau}$$

so that,

$$\begin{aligned}\tau_1 &= \tau_0 + \overline{\tau} + \Delta_1 \\ &= \hat{\tau}_0 + \overline{\tau} + \Delta_1 - \xi_{\tau_0}\end{aligned}$$

where ξ_{τ_0} is the error in estimating τ_0. Because ξ_{τ_0} is a function of ε_0 it follows that

$$\begin{aligned}\mathrm{Var}[\Delta_1 - \xi_{\tau_0}] &= \sigma^2_{\Delta_1} + \sigma^2_{\xi_{\tau_0}} \\ &= 2\,\sigma_0^2 + \sigma^2_{\xi_{\tau_0}}\end{aligned}$$

where the last line follows from (4.3.1c). Therefore, given the estimate τ_0, the mean and variance of τ_1 are just:

$$E[\tau_1] = \hat{\tau}_0 + \overline{\tau}$$

$$\sigma^2_{\tau_1} = \mathrm{Var}[\tau_1] = 2\,\sigma_0{}^2 + \sigma^2_{\xi_{\tau_0}}$$

The above may now be regarded as a priori statistics. Then, using the results in Appendix H, the observation (4.5.8) leads to the estimate

$$\hat{\tau}_1 = \gamma_1 1'(I-F_1(F_1'F_1)^{-1}F_1')\ [\hat{T}_1-k_1-(\hat{\tau}_0+\bar{\tau})1] + \hat{\tau}_0+\bar{\tau} \qquad (4.5.10)$$

where

$$\gamma_1 = \frac{2\sigma_0^{\ 2} + \sigma^2_{\xi_{\tau_0}}}{\sigma^2+(2\sigma_0^{\ 2}+\sigma^2_{\xi_{\tau_0}})(K-1'F_1(F_1'F_1)^{-1}F_1'1)} \qquad (4.5.11)$$

Denoting the error in this estimate as

$$\xi_{\tau_1} = \hat{\tau}_1 - \tau_1$$

we have, from Appendix H, that

$$\sigma^2_{\xi_{\tau_1}} = \frac{\sigma^2(2\sigma_0^{\ 2} + \sigma^2_{\xi\tau_0})}{\sigma^2+(2\sigma_0^2+\sigma^2_{\xi_{\tau_0}})(K-1'F_1(F_1'F_1)^{-1}F_1'1)} = \sigma^2\gamma_1 \qquad (4.5.12)$$

Having estimated τ_1, we can now estimate p_1. From Appendix H we get the following:

$$\hat{p}_1 = p_1^* + (F_1'F_1)^{-1}F_1'(\hat{T}_1-k_1-\hat{\tau}_1 1) \qquad (4.5.13)$$

and the error covariance matrix and rms error are

$$P_{\xi_{p_1}} = \sigma^2(F_1'F_1)^{-1} + \sigma^2_{\xi_{\tau_1}}(F_1'F_1)^{-1}F_1'11'F_1(F_1'F_1)^{-1} \tag{4.5.14}$$

$$\mathrm{Var}[\|\xi_{p_1}\|] = \sigma^2\ \mathrm{tr}\{(F_1'F_1)^{-1}\} + \sigma^2_{\xi_{\tau_1}} 1'F_1(F_1'F_1)^{-2}F_1'1 \tag{4.5.15}$$

Formulas (4.5.10) through (4.5.15) exhibit a marked similarity to (4.5.2) through (4.5.7). Note in particular, that the estimates for τ_0 and τ_1 are used in exactly the same manner in finding the estimates for p_0 and p_1, respectively.

We are now in a position to determine how $\hat{\tau}_n$ is to be computed from $\hat{T}_n$ and $\hat{\tau}_{n-1}$. It is clear that, given $\hat{\tau}_{n-1}$, the mean of τ_n is

$$E[\tau_n|\hat{\tau}_{n-1}] = \hat{\tau}_{n-1} + \overline{\tau}$$

Also, let us define the variance of τ_n, given $\hat{\tau}_{n-1}$, as $\sigma^2_{\tau_n}$:

$$\sigma^2_{\tau_n} \triangleq \mathrm{Var}[\tau_n|\hat{\tau}_{n-1}]$$

Then, from Appendix H we have

$$\hat{\tau}_n = \gamma_n 1'[I-F_n(F_n'F_n)^{-1}F_n'][\hat{T}_n-k_n-(\hat{\tau}_{n-1}+\overline{\tau})1] + \hat{\tau}_{n-1} + \overline{\tau} \tag{4.5.16}$$

$$\hat{p}_n = p^*_n + (F_n'F_n)^{-1}F_n'(\hat{T}_n-k_n-\hat{\tau}_n 1) \tag{4.5.17}$$

where

$$\gamma_n = \frac{\sigma^2_{\tau_n}}{\sigma^2+\sigma^2_{\tau_n}(K-1'F_n(F_n'F_n)^{-1}F_n'1)}$$

The resulting error statistics are just

$$\sigma^2_{\xi_{\tau_n}} = \sigma^2 \gamma_n \tag{4.5.18}$$

$$P_{\xi_{p_n}} = \sigma^2(F_n'F_n)^{-1} + \sigma^2_{\xi_{\tau_n}} (F_n'F_n)^{-1}F_n'11'F_n(F_n'F_n)^{-1} \tag{4.5.19}$$

Thus, the estimates $\hat{\tau}_n$ and $\hat{p}_n$ are well-defined by (4.5.16) and (4.5.17) if we can determine the constants γ_n. For notational simplicity we define

$$\beta_n = K-1'F_n(F_n'F_n)^{-1}F_n'1$$

so that

$$\gamma_n = \frac{\sigma^2_{\tau_n}}{\sigma^2+\beta_n\sigma^2_{\tau_n}} \tag{4.5.20}$$

To find $\sigma^2_{\tau_n}$ we write

$$\tau_n = \tau_{n-1} + \bar{\tau} + \Delta_n$$

$$= \hat{\tau}_{n-1} + \bar{\tau} + \Delta_n - \xi_{\tau_{n-1}}$$

Therefore,

$$\sigma^2_{\tau_n} = \mathrm{Var}[\Delta_n - \xi_{\tau_{n-1}}]$$

$$= \mathrm{Var}[\Delta_n] + \mathrm{Var}[\xi_{\tau_{n-1}}] - 2E[\Delta_n \xi_{\tau_{n-1}}]$$

$$= 2\sigma_o^2 + \sigma^2 \gamma_{n-1} - 2E[\Delta_n \xi_{\tau_{n-1}}] \qquad (4.5.21)$$

where the last line follows from (4.3.1) and (4.5.18).

To determine $E[\Delta_n \xi_{\tau_{n-1}}]$ we need an expression for $\xi_{\tau_{n-1}}$. From (4.5.16),

$$\xi_{\tau_{n-1}} = \hat{\tau}_{n-1} - \tau_{n-1}$$

$$= \gamma_{n-1} 1'[I - F_{n-1}(F'_{n-1}F_{n-1})^{-1}F'_{n-1}][\hat{T}_{n-1} - k_{n-1} - (\hat{\tau}_{n-2} + \bar{\tau})1] + \hat{\tau}_{n-2} + \bar{\tau} - \tau_{n-1}$$

But,

$$\hat{\tau}_{n-2} + \bar{\tau} - \tau_{n-1} = \xi_{\tau_{n-2}} - \Delta_{n-1}$$

and, using (4.5.8),

$$\xi_{\tau_{n-1}} = \gamma_{n-1} 1'[I - F_{n-1}(F'_{n-1}F_{n-1})^{-1}F'_{n-1}][F_{n-1}(p_{n-1} - p^*_{n-1}) - (\xi_{\tau_{n-2}} - \Delta_{n-1})1 + \varepsilon_{n-1}]$$

$$+ \xi_{\tau_{n-2}} - \Delta_{n-1}$$

$$\xi_{\tau_{n-1}} = (\gamma_{n-1}\beta_{n-1} - 1)(\Delta_{n-1} - \xi_{\tau_{n-2}}) + \text{term in } \varepsilon_{n-1}$$

Therefore, using (4.5.20)

$$E[\Delta_n \xi_{\tau_{n-1}}] = -\frac{\sigma^2}{\sigma^2+\beta_{n-1}\sigma^2_{\tau_{n-1}}} \left(E[\Delta_n\Delta_{n-1}] - E[\Delta_n \xi_{\tau_{n-2}}]\right) \qquad (4.5.22)$$

From (4.3.1) we can compute $E[\Delta_n\Delta_{n-i}]$ recursively as follows:

$$E[\Delta_n{}^2] = 2\sigma_o{}^2$$

$$E[\Delta_n\Delta_{n-1}] = E[(\Delta_{n-1}+\Delta_n)\Delta_{n-1}-\Delta_{n-1}^2]$$

$$= (\sigma_o{}^2+2\sigma_o{}^2) - 2\sigma_o{}^2 = \sigma_o{}^2$$

$$E[\Delta_n\Delta_{n-2}] = E[(\Delta_{n-2}+\Delta_{n-1}+\Delta_n)\Delta_{n-2}-\Delta_{n-2}^2-\Delta_{n-1}\Delta_{n-2}]$$

$$= (\sigma_o{}^2+3\sigma_o{}^2) - 2\sigma_o{}^2-\sigma_o{}^2 = \sigma_o{}^2$$

and, in general,

$$E[\Delta_n\Delta_{n-i}] = \begin{cases} 2\sigma_o^2, & i=0 \\ \sigma_o{}^2, & 1\leq i\leq n-1 \end{cases} \qquad (4.5.23)$$

We can now use (4.5.20) through (4.5.23) to construct the sequences $\{\gamma_n\}$ and $\{\sigma^2_{\tau_n}\}$. We have, from (4.5.2)

$$\gamma_o = \frac{1}{\beta_o}$$

Then,

$$\sigma^2_{\tau_1} = 2\sigma_0{}^2 + \sigma^2\gamma_0$$

$$\gamma_1 = \frac{\sigma^2_{\tau_1}}{\sigma^2+\beta_1\sigma^2_{\tau_1}}$$

$$\sigma^2_{\tau_2} = 2\sigma_0{}^2 + \sigma^2\gamma_1 + \frac{2\sigma^2}{\sigma^2+\beta_1\sigma^2_{\tau_1}} E[\Delta_2\Delta_1]$$

$$= 2\sigma_0{}^2 + \frac{\sigma^2\sigma^2_{\tau_1}}{\sigma^2+\beta_1\sigma^2_{\tau_1}} + \frac{2\sigma^2\sigma^2_0}{\sigma^2+\beta_1\sigma^2_{\tau_1}}$$

$$\gamma_2 = \frac{\sigma^2_{\tau_2}}{\sigma^2+\beta_2\sigma^2_{\tau_2}}$$

$$\sigma^2_{\tau_3} = 2\sigma_0{}^2 + \sigma^2\gamma_2 + \frac{2\sigma^2}{\sigma^2+\beta_2\sigma^2_{\tau_2}}(E[\Delta_3\Delta_2] + \frac{\sigma^2}{\sigma^2+\beta_1\sigma^2_{\tau_1}}E[\Delta_3\Delta_1])$$

$$= 2\sigma_0{}^2 + \frac{\sigma^2\sigma^2_{\tau_2}}{\sigma^2+\beta_2\sigma^2_{\tau_2}} + \frac{2\sigma^2\sigma_0{}^2}{\sigma^2+\beta_2\sigma^2_{\tau_2}}(1 + \frac{\sigma^2}{\sigma^2+\beta_1\sigma^2_{\tau_1}})$$

Continuing in this manner, it is apparent that

$$\sigma^2_{\tau_n} = \frac{\sigma^2\sigma^2_{\tau_{n-1}}}{\sigma^2+\beta_{n-1}\sigma^2_{\tau_{n-1}}}$$

$$+ 2\sigma_o{}^2\left[1 + \frac{\sigma^2}{\sigma^2+\beta_{n-1}\sigma^2_{\tau_{n-1}}}\left[1 + \frac{\sigma^2}{\sigma^2+\beta_{n-2}\sigma^2_{\tau_{n-2}}}\left[1 + \ldots\right.\right.\right.$$

$$\left.\left.\ldots\left[1 + \frac{\sigma^2}{\sigma^2+\beta_1\sigma^2_{\tau_1}}\right]\ldots\right] \qquad (4.5.24)$$

and

$$\gamma_n = \frac{\sigma^2_{\tau_n}}{\sigma^2+\beta_n\sigma^2_{\tau_n}} \qquad (4.5.25)$$

The constants β_n depend on the unit vectors from the satellites to the aircraft:

$$\beta_n = K-1'F_n(F_n'F_n)^{-1}F_n'1$$

where

$$F_n = \frac{1}{c}\begin{pmatrix} u'_{n1} \\ \vdots \\ u'_{nK} \end{pmatrix}$$

An arbitary rotation of all of these unit vectors will carry F_n to the matrix $\overline{F}_n$:

$$\overline{F}_n = F_n R'$$

where R is an orthogonal matrix:

$$R' = R^{-1}$$

Under this transformation, β_n changes to $\overline{\beta}_n$:

$$\overline{\beta}_n = K\text{-}1'\overline{F}_n(\overline{F}'_n\overline{F}_n)^{-1}\overline{F}'_n 1$$

$$= K\text{-}1'F_nR'(RF'_nF_nR')^{-1}RF'_n 1$$

$$= K\text{-}1'F_nR'R(F'_n F_n)^{-1}R'RF'_n 1$$

$$= \beta_n$$

Since over a short period of time there will be only slight changes in the angles between the unit vectors, we may assume that

$$\beta_n = \beta = \text{constant}$$

With this assumption, it is easy to argue that the numbers $\sigma^2_{\tau_n}$ converge to a limit σ^2_τ. In fact, at this limit, we have from (4.5.24)

$$\sigma_\tau^2 = \frac{\sigma^2\sigma_\tau^2}{\sigma^2+\beta\sigma_\tau^2} + 2\sigma_o^2 \sum_{i=0}^{\infty}\left[\frac{\sigma^2}{\sigma^2+\beta\sigma_\tau^2}\right]^i$$

Then, since from (4.5.25) we have

$$\sigma_\tau^2 = \frac{\gamma\sigma^2}{1-\gamma\beta}$$

$$\frac{2}{\sigma^2+\beta\sigma_\tau^2} = 1-\gamma\beta$$

where γ is the limit of the γ_n, we now have the following equation in γ :

$$\frac{\gamma\sigma^2}{1-\gamma\beta} = \gamma\sigma^2 + 2\,\sigma_o^2\sum_{i=0}^{\infty}[1-\gamma\beta]^i$$

$$= \gamma\sigma^2 + \frac{2\sigma_o^2}{\gamma\beta}$$

Therefore, γ satisfies the following cubic equation:

$$\gamma^3 + \frac{2\sigma_o^2}{\sigma^2\beta}\gamma - \frac{2\sigma_o^2}{\sigma^2\beta^2} = 0$$

Using the well-known method for finding roots of cubic equations [7, pg. 7], it is easy to show that the above cubic equation has only one real root. The value of this root is

$$\gamma = \left(\frac{\sigma_o^2}{\sigma^2\beta^2}\right)^{1/3} \left\{\left[1+\sqrt{1+\frac{8}{27}\frac{\sigma_o^2\beta}{\sigma^2}}\right]^{1/3} + \left[1-\sqrt{1+\frac{8}{27}\frac{\sigma_o^2\beta}{\sigma^2}}\right]^{1/3}\right\} \qquad (4.5.26)$$

where

$$\beta = K - 1'F(F'F)^{-1}F'1$$

We may now note that, rather than use the numbers γ_n as given by (4.5.24) and (4.5.25), we could probably use γ, as given in (4.5.26), in determining the estimates $\hat{\tau}_n$ and $\hat{p}_n$ by (4.5.16) and (4.5.17). The only difference would be that the estimates for small values of n would not be quite as good.

4.5.3 Tracking $\{p_n\}$ for $n \geq N+1$ with Three Satellites

Suppose that we have tracked $\{\tau_n\}$ and $\{p_n\}$ with four or more satellites, and that the estimators have reached steady state by time N, i.e., that

$$\gamma_N = \gamma$$

where γ is given by (4.5.26). Also, suppose that between times N and N+1 the number of available satellites drops to three. For example, a satellite might fail or become disadvantaged due to aircraft banking. We would like to determine how long the aircraft may be tracked with acceptable error using only three satellites.

Since we have only three satellites, it follows from Appendix H that the best estimate for τ_n is just the mean of τ_n given $\hat{\tau}_N$. But

$$\tau_n = \tau_N + (n-N)\bar{\tau} + \sum_{i=N+1}^{n} \Delta_i$$

$$= \hat{\tau}_N + (n-N)\bar{\tau} + \sum_{i=N+1}^{n} \Delta_i - \xi_{\tau_N}$$

Therefore,

$$\hat{\tau}_n = \hat{\tau}_N + (n-N)\bar{\tau}$$

and

$$\sigma^2_{\xi_{\tau_N}} = \mathrm{Var}\left[\sum_{i=N+1}^{n} \Delta_i - \xi_{\tau_N}\right]$$

$$= \mathrm{Var}\left[\sum_{i=N+1}^{n} \Delta_i\right] + \sigma^2_{\xi_{\tau_N}} - 2E\left[\sum_{i=N+1}^{n} \Delta_i \xi_{\tau_N}\right]$$

From (4.3.1) we have

$$\mathrm{Var}\left[\sum_{i=N+1}^{n} \Delta_i\right] = \sigma_o{}^2[(n-N) + (n-N)^2]$$

and from (4.5.18)

$$\sigma^2_{\xi_{\tau_N}} = \sigma^2{}_Y$$

From the previous section, for i > N,

$$- E[\Delta_i \xi_{\tau_N}] = \sum_{j=1}^{\infty} \sigma_o^2 (1-\gamma\beta)^j$$

$$= \sigma_o^2 \frac{1-\gamma\beta}{\gamma\beta}$$

Thus,

$$\sigma^2_{\xi_{\tau_n}} = \sigma^2\gamma + \sigma_o^2 \left[(n-N) \left(\frac{2-2\gamma\beta}{\gamma\beta} + 1\right) + (n-N)^2\right]$$

$$\sigma^2_{\xi_{\tau_n}} = \sigma^2\gamma + \sigma_o^2 \left[\frac{2-\gamma\beta}{\gamma\beta} (n-N) + (n-N)^2\right] \qquad (4.5.27)$$

Then, from Appendix H, the estimate for $\hat{p}_n$ is

$$\hat{p}_n = p_n^* + F_n^{-1}(\hat{T}_n - k_n - \hat{\tau}_n 1) \qquad (4.5.28)$$

and the error statistics are

$$P_{\xi_{p_n}} = \sigma^2 (F_n' F_n)^{-1} + \sigma^2_{\xi_{\tau_n}} F_n^{-1} 1 1' F_n'^{-1} \qquad (4.5.29)$$

$$\text{Var}\,[\|\xi_{p_n}\|] = \sigma^2 \,\text{tr}\{(F_n' F_n)^{-1}\} + \sigma^2_{\xi_{\tau_n}} 1'(F_n' F_n)^{-1} 1 \qquad (4.5.30)$$

where, in these formulas we have used the fact that since F_n is now 3 by 3, F_n^{-1} exists if $(F_n'F_n)^{-1}$ does.

Applying (4.5.27), we see that $\mathrm{Var}[\|\xi_{p_n}\|]$ increases as $(n-N)^2$ for large values of (n-N):

$$\mathrm{Var}\,[\|\xi_{p_n}\|] = \sigma^2\,\mathrm{tr}\{(F_n'F_n)^{-1}\}$$

$$+ \{\sigma^2\gamma + \sigma_0^{\,2}[\tfrac{2-\gamma\beta}{\gamma\beta}(n-N) + (n-N)^2]\}1'(F_n'F_n)^{-1}\,1$$

In the above, of course, γ and β are the limiting value of γ_n and the constant value of β_n, respectively, for $0<n\leq N$.

4.5.4 Summary of Tracking and Estimating Procedures

Probably the most striking observation to be made from the results of the previous three sections is the form of the estimate of p_n. In all three cases, we have

$$\hat{p}_n = p_n^* + (F_n'F_n)^{-1}F_n'[\hat{T}_n - k_n - \hat{\tau}_n 1]$$

where $\hat{\tau}_n$ is the appropriate estimate of τ_n. In the special case where K=3, we can replace the matrix $(F_n'F_n)^{-1}F_n'$ by the matrix F_n^{-1}, since F_n is square in this case. Of course, we have assumed throughout that $(F_n'F_n)^{-1}$ exists;

this is equivalent to assuming that three of the unit vectors u_{ni}, i=1,2,...K, are linearly independent.

The method of estimating τ_n does vary, however. We use a least squares estimate of τ_n whenever $K \geq 4$ and we have no statistical information, or at best very poor statistical information, about τ_n. This will be true when n=0, or just after having tracked $\{\tau_n\}$ for some time with K=3. This least squares estimate, for the case n=0, takes the form

$$\hat{\tau}_0 = \gamma_0 1'(I-F_0(F_0'F_0)^{-1}F_0')(\hat{T}_0-k_0)$$

where

$$\gamma_0 = \frac{1}{\beta_0} = \frac{1}{K-1'F_0(F_0'F_0)^{-1}F_0'1}$$

It is important to notice that if K=3, then β_0 is

$$\beta_0 = K-1'F_0(F_0'F_0)^{-1}F_0'1$$

$$= K-1'1 = 0$$

Thus, it is impossible to estimate τ_0 with less than four satellites.

For $n \geq 1$, and with $K \geq 4$, the estimates $\hat{\tau}_0$, $\hat{\tau}_1, \ldots \hat{\tau}_{n-1}$ provide statistical information about τ_n. Thus, we use a constrained linear minimum-mean-square-error estimate of τ_n; this estimate is of the form

$$\hat{\tau}_n = \gamma_n 1'[I-F_n(F_n'F_n)^{-1}F_n'][\hat{T}_n-k_n-(\hat{\tau}_{n-1}+\bar{\tau})1] + \hat{\tau}_{n-1} + \bar{\tau}$$

where

$$\gamma_n = \frac{\sigma^2_{\tau_n}}{\sigma^2+\sigma^2_{\tau_n}\beta_n} \tag{4.5.31}$$

$$= \frac{\sigma^2_{\tau_n}}{\sigma^2+\sigma^2_{\tau_n}(K-1'F_n(F_n'F_n)^{-1}F_n'1)}$$

and the sequence $\{\sigma^2_{\tau_n}\}$ is given by (4.5.24) with

$$\sigma^2_{\tau_1} = 2\sigma_o^2 + \sigma^2\gamma_o$$

It was argued that β_n can be expected to be almost constant over fairly long periods of time. The criteria for β_n to be constant are just that the relative angles between the unit vectors u_{ni} remain constant, and that K remains constant. Under this assumption it was shown that the sequence $\{\gamma_n\}$ converges to a limit γ given by (4.5.26). Defining the parameter α as

$$\alpha = \frac{8}{27}\frac{\sigma_o^2\beta}{\sigma^2} \tag{4.5.32}$$

where β is the constant value of β_n, (4.5.26) can be written as

$$\gamma = \frac{1}{\beta}\,\psi(\alpha) \tag{4.5.33}$$

where

$$\psi(\alpha) = \frac{3}{2}\,\alpha^{1/3}\{[1+\sqrt{1+\alpha}]^{1/3} + [1-\sqrt{1+\alpha}]^{1/3}\} \tag{4.5.34}$$

Finally, if K suddenly drops to three, due to aircraft banking or a satellite failure, we must estimate τ_n by extrapolating from the estimates $\hat{\tau}_o$, $\hat{\tau}_1$, $\ldots\hat{\tau}_N$, where N is the last time that four or more satellites are available. The estimate is simply

$$\hat{\tau}_n = \hat{\tau}_N + (n-N)\overline{\tau}$$

We are also interested in the variance of the norm of the position estimate error, ξ_{p_n}. From (4.5.5) and (4.5.7), (4.5.18) and (4.5.19), and (4.5.27) and (4.5.30) we have

$$\text{Var}\,[\|\xi_{p_n}\|] = \sigma^2[\text{tr}\{(F_n{}'F_n)^{-1}\} + \gamma_n 1'F_n(F_n{}'F_n)^{-2}F_n{}'1] \tag{4.5.35}$$

for all three cases. Recall that σ^2 is the variance in measuring the time of arrival T_{ni}, plus $\frac{1}{c^2}$ times the variance of the tracking error, in the radial direction, of satellite i, plus the variance of random relays in the signal propagation due to atmosphere refraction.

The parameters γ_n in (4.5.35) depend on the type of estimate that is used for τ_n. When we use the least squares estimate, e.g., when n=0,

$$\gamma_0 = \frac{1}{\beta} \tag{4.5.36}$$

When we are tracking the sequence $\{\tau_n\}$ with a linear estimator, γ_n is given by (4.5.30), and we have seen that γ_n converges to γ:

$$\gamma_n \to \gamma = \frac{1}{\beta}\psi(\alpha) \tag{4.5.37}$$

For the case where K=3 and we are estimating τ_n by extrapolation from $\hat{\tau}_N$, we have from (4.5.27), (4.5.32), and (4.5.33)

$$\gamma_n = \frac{1}{\beta}[\psi(\alpha) + \frac{27}{8}\,\alpha\{\frac{2-\psi(\alpha)}{\psi(\alpha)}\,(n-N) + (n-N)^2\}]$$

In the above expressions we are assuming that K remains constant for $0 \leq n \leq N$, and that $\beta_n = \beta$ for $0 \leq n \leq N$.

We have shown in Section 4.5.3 that if K is constant for $0 \leq n \leq N$ and if the angles between the unit vectors u_{ni} are constant over this interval, then β_n is constant for $0 \leq n \leq N$. A similar argument shows that under these assumptions the quantities

$$tr\{(F_n'F_n)^{-1}\}$$

and

$$1'F_n(F_n'F_n)^{-2}F_n'1$$

also remain constant for $0 \leq n \leq N$. Therefore, the quantity $\psi(\alpha)$ in (4.5.37) is just the factor by which the second term in (4.5.35) is reduced, from its value at n=0 to its limiting value in the interval $0<n \leq N$, by tracking the sequence $\{\tau_n\}$. We shall see from examples in the next section that the second term in (4.5.35) always significantly dominates the first term; thus, by tracking the sequence $\{\tau_n\}$, $Var[\|\xi_{p_n}\|]$ can be reduced to approximately $\psi(\alpha) \cdot Var[\|\xi_{p_o}\|]$.

The quantity $\psi(\alpha)$ does not have a simple interpretation in terms of the relative sizes of $Var[\|\xi_{p_o}\|]$ and $Var[\|\xi_{p_n}\|]$ for n>N. The reason is that when the number of satellites drops to three, the quantities tr $\{(F_n'F_n)^{-1}\}$ and $1'F_n(F_n'F_n)^{-2}F_n'1$ both increase, but by amounts that depend on the particular satellite constellation.

The value of $\psi(\alpha)$ does, however, play a role in how quickly the sequence $\{\gamma_n\}$, for $0<n \leq N$, converges to its limiting value γ. In fact, from the derivation of (4.5.26) it is apparent that the rapidity with which γ_n converges to γ is more or less governed by how quickly the geometric series

$$s_n = \sum_{i=0}^{n} (1-\gamma\beta)^i = \frac{1-[1-\psi(\alpha)]^{n+1}}{\psi(\alpha)}$$

converges to the limit $[\psi(\alpha)]^{-1}$. For example, we can say that the number of steps necessary for γ_n to converge to within $(0.1)\gamma$ of γ is approximately

$$n \cong \frac{1}{\log(1-\psi(\alpha))} - 1$$

$$\cong \frac{2.3}{\psi(\alpha)} - 1$$

for small values of $\psi(\alpha)$. Thus, we would like $\psi(\alpha)$ to be small, so that $\mathrm{Var}[\|\xi_{p_n}\|]$ is small for $0<n\leq N$, but not so small that the convergence of $\{\gamma_n\}$ to γ is very slow.

A plot of the function $\psi(\alpha)$ is shown in Figure 4.1. Two useful approximations may be derived from (4.5.34); they are

$$\psi(\alpha) \cong 1-\frac{4}{27}\alpha^{-1}, \text{ for } \alpha\geq 1$$

$$\psi(\alpha) \cong \frac{3}{2}(2)^{1/3}[\alpha^{1/3}-(\frac{\alpha}{2})^{2/3}]$$

$$\cong 1.89[\alpha^{1/3}-(\frac{\alpha}{2})^{2/3}], \text{ for } \alpha\leq 0.3$$

These approximations also appear on Figure 4.1.

4.6 EXAMPLES

We shall now consider the two satellite constellations from Appendix D of reference 1. For these two constellations we can determine which values of τ_0 lead to significant improvements in tracking the aircraft as a result of also tracking the sequence $\{\tau_n\}$. Also, we can determine how long the aircraft may be tracked using only three satellites. These illustrated examples provide considerable insight on the role of tracking.

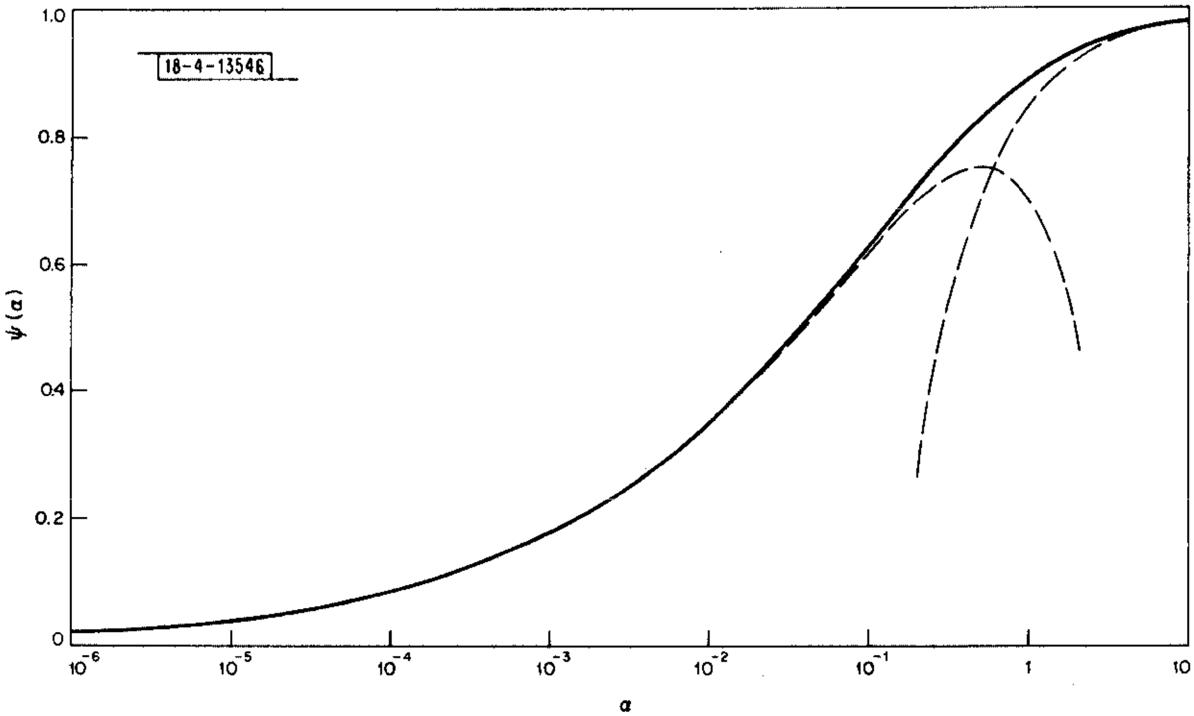

Fig. 4.1. The function $\psi(\alpha)$ and two approximations.

In the following two examples we shall assume that

$$\sigma^2 = (50 \text{ nsec})^2$$

where σ^2 is the variance is the estimate of time of arrival at any particular satellite. In each of these examples we shall consider only three values for σ_0^2: $(5 \text{ nsec})^2$, $50 \text{ nsec})^2$, and $500 \text{ nsec})^2$. It will be seen that the point at which one begins to obtain only marginal gains by tracking $\{\tau_n\}$ is the point at which

$$\sigma_0^2 \cong \sigma^2$$

In each of the two examples there are initially seven or eight satellites, all within a 45° half-angle cone with a vertical axis and the vertex at the aircraft. This restriction, i.e., all satellites lying within such a cone, was used in reference 1 to insure that the signal from an aircraft, banking 30° in any direction, could always be received by each of the satellites at a level sufficient for detection.

In determing how long the aircraft can be tracked using only three satellites, the following method was used. The three most optimally positioned satellites of the initial seven or eight were selected; these satellites were the most widely spaced triplet of satellites within the constraint cone. It was then assumed that at time n=N, all satellites but these three were disadvantaged. The results on tracking the aircraft with three satellites are optimistic, in the sense that the three best satellites are used.

The latitudes and longitudes of the subsatellite points for the two examples are listed in Table 4.1. The satellite numbering is such that satellites 1, 2, and 3 are used in tracking with three satellites. In both examples, the aircraft is at

aircraft latitude = 45°

aircraft longitude = 120°

The parameters β, $tr\{(F'F)^{-1}\}$, and $1'F(F'F)^{-2}F'1$ are listed in Table 4.2. It should be noted that in both examples, β is quite small. Thus, in the expression for the variance of the least-squares estimate for p_o:

$$Var[\| \xi_{p_o} \|] = \sigma^2[tr\{(F'F)^{-1}\} + \frac{1}{\beta}1'F(F'F)^{-2}F'1]$$

the second term obviously dominates. This is particularly true in Example II. It should also be noted that when the number of satellites is reduced to three, the quantities $tr\{(F'F)^{-1}\}$ and $1'F(F'F)^{-2}F'1$ both increase; this increase is substantial in Example II.

For each of the examples, and for the three values of σ_o: 5 nsec, 50 nsec, and 500 nsec, the following quantities were determined:

Table 4.1. Latitudes and Longitudes of Sub-Satellite Points.

Example	Satellite	Latitude	Longitude
I	1	14.9	152.7
	2	63.4	115.0
	3	36.6	101.8
	4	51.0	123.2
	5	51.0	106.8
	6	60.1	80.5
	7	60.1	79.5
II	1	27.0	109.8
	2	44.5	129.5
	3	62.6	80.4
	4	62.6	114.6
	5	56.2	111.6
	6	27.0	85.3
	7	44.5	65.5
	8	56.2	83.4

Table 4.2. Satellite Constellation Parameters.

		$tr\{(F'F)^{-1}\}$		$1'F(F'F)^{-2}F'1$	
Example	β	All Satellites	Satellites 1,2,3	All Satellites	Satellites 1,2,3
I	.0543	7.84	9.39	1.30	5.76
II	.00355	3.99	14.9	1.16	14.3

(i) α (Eq. (4.5.32))

(ii) $\psi(\alpha)$ (Fig. 4.1)

(iii) $\sqrt{\mathrm{Var}[\|\xi_{p_o}\|]}$ = standard deviation for least squares estimate

(iv) $\sqrt{\mathrm{Var}[\|\xi_{p_N}\|]}$ = limiting standard deviation while tracking $\{\tau_n\}$ with all satellites

(v) $\sqrt{\mathrm{Var}[\|\xi_{p_{N+1}}\|]}$ = standard deviation of first estimate made with three satellites.

These quantities are exhibited in Table 4.3. Finally, Figures 4.2 and 4.3 illustrate how $\sqrt{\mathrm{Var}[\|\xi_{p_n}\|]}$ varies with n.

Let us denote, as in subsection 4.1, the geometric dilution, using the least squares estimate, as

$$k = \frac{1}{\sigma}\sqrt{\mathrm{Var}[\|\xi_{p_o}\|]}$$

$$= \sqrt{\mathrm{tr}\{(F'F)^{-1}\} + \frac{1}{\beta}\, 1'F(F'F)^{-2}F'1}$$

then, examination of Table 4.3 and Figures 4.2 and 4.3 reveals the following rules of thumb:

(1) When β is very small, and thus k very large, (Example II) tracking $\{\tau_n\}$ with all satellites can significantly reduce $\sqrt{\mathrm{Var}[\|\xi_{p_n}\|]}$ if $\sigma_o \lesssim \sigma$.

(2) When k is not particularly large (Example I), tracking $\{\tau_n\}$ with all satellites will reduce $\sqrt{\mathrm{Var}[\|\xi_{p_n}\|]}$, but this reduction is significant only when $\sigma_o \lesssim .1\sigma$.

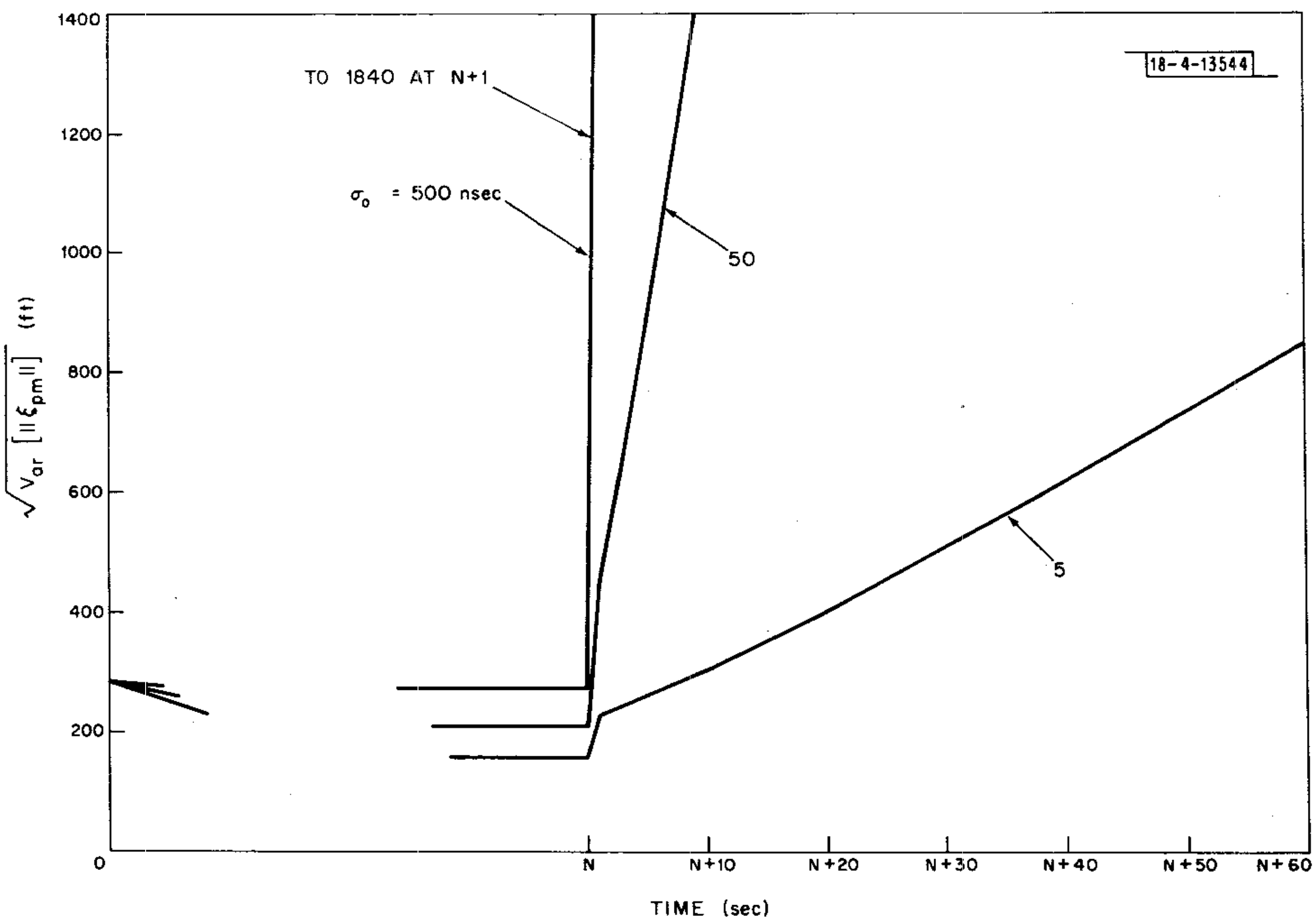

Fig. 4.2. RMS error for Example I.

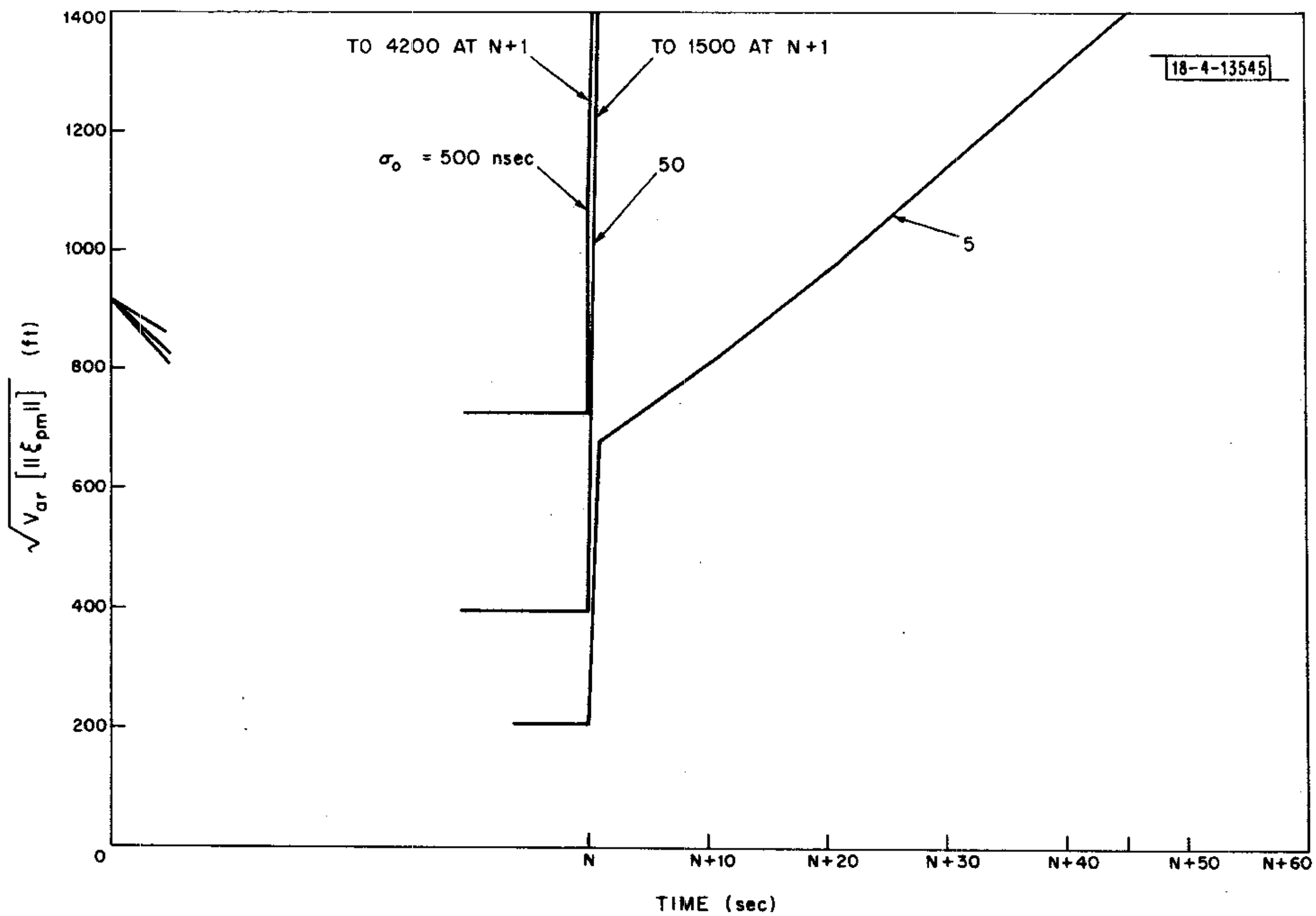

Fig. 4.3. RMS error for Example II.

Table 4.3. Estimation Parameters.

Example	σ	σ_o	α	$\psi(\alpha)$	$\sqrt{Var[\|\xi_{p_0}\|]}$	$\sqrt{Var[\|\xi_{p_N}\|]}$	$\sqrt{Var[\|\xi_{p_{N+1}}\|]}$
I	50 nsec	5 nsec	1.61×10^{-4}	.10	282 ft	160 ft	230 ft
	50 nsec	50 nsec	1.61×10^{-2}	.40	282 ft	209 ft	450 ft
	50 nsec	500 nsec	1.61	.925	282 ft	274 ft	1840 ft
II	50 nsec	5 nsec	1.05×10^{-5}	.04	910 ft	205 ft	677 ft
	50 nsec	50 nsec	1.05×10^{-3}	.18	910 ft	396 ft	1500 ft
	50 nsec	500 nsec	1.05×10^{-1}	.635	910 ft	727 ft	4200 ft

(3) There is a significant jump between $\sqrt{Var[\|\xi_{p_N}\|]}$ and $\sqrt{Var[\|\xi_{p_{N+1}}\|]}$. This is due largely to the increases in $tr\{(F'F)^{-1}\}$ and $1'F(F'F)^{-2}F'1$ when the number of satellites drops to three (Table 4.2).

(4) If $\sigma_0 \tilde{<} .1\ \sigma$, then the aircraft may be tracked using three satellites. The interval of time over which the tracking error will be acceptable varies from about 30 to 60 seconds.

4.7. CONCLUSIONS

The examples of the preceeding section indicate that the gains to be derived by tracking the sequence $\{\tau_n\}$ are significant only when the accuracy of the aircraft clock is greater than or equal to the accuracy with which the pulse times of arrival are measured, i.e., when $\sigma_0 \leq \sigma$. We have seen that if $\sigma_0 \cong \sigma$, then the aircraft position estimates may be improved by tracking $\{\tau_n\}$, when four or more satellites are used; this improvement may be quite significant when the geometric dilution with no tracking is large. However, it is only when $\sigma_0 \cong 0.1\sigma$ that one can estimate aircraft positions, with acceptable error, over several tens of seconds, while using only three satellites.

The above conclusions result, at least in part, from two assumptions that we have made, one concerning the model for $\{\tau_n\}$ and the other concerning the form of the estimator for τ_n. The first of these assumptions was that (see Appendix H and subsection 4.3) $\sigma_1^2 = \sigma_2^2$, that the component of $Var[\tau_n]$ due to the independent increment process and the component of $Var[\tau_n]$ due to "1/f" noise are comparable, when n=1. This assumption seems to be in agreement with the specifications of typical oscillators. However, it could be that the value of n, where these two components are comparable, is much larger, say, n=100. If this were the case, then the improvements to be gained by tracking $\{\tau_n\}$ would be more significant, particularly when tracking the aircraft with only three satellites.

The second assumption was that a reasonably good estimate of τ_n could be obtained even if it were constrained to be a function of only $\hat{\tau}_{n-1}$, the most recent estimate, and $\hat{T}_n$, the current measurements of times of arrival. The estimator that was derived is the optimal linear estimator subject to this constraint. However, it may be easily shown that $\{\tau_n\}$ is not a Markov process of finite order when $\frac{1}{2}<\rho<1$ (see Appendix G for the significance of ρ); thus, in particular, the estimate $\hat{\tau}_{n-1}$ does not contain all of the relevant information in the sequence $\{\hat{T}_i, i\leq n-1\}$, as it would if $\{\tau_n\}$ were a first order Markov process. Only additional analysis will reveal if a significantly more nearly optimal estimator would result if τ_n were allowed to be a function of, say, $\hat{\tau}_{n-M}$, $\hat{T}_{n-M+1}, \ldots \hat{T}_n$ for some small integer M>1.

APPENDIX A

THE DEPENDENCE OF P_D ON p_d

In this appendix we derive a lower bound to the parameter P_D, which illustrates its dependence upon the threshold decision rule parameter, p_d. In order to derive this lower bound we must specify the procedure by which the first stage decision is carried out. We now do this.

The first stage decision decides whether or not a given aircraft is in the airspace. Consider a particular aircraft, aircraft "i". The first stage decision rule that we shall assume will depend upon the number of lists in the set {List (i,j), j=1,...K} which have at least one entry. Of course the representative surveillance system is not forced to operate with this decision procedure. However, this is a procedure which is available to it and it does use most of the data available to the ground processor in an optimum way.

Let us define the following indicator function:

$$\rho\,(i,j) = 1, \quad \text{if List } (i,j) \text{ has at least one entry} \qquad (A.1)$$
$$= 0, \text{ otherwise}$$

Let us also assume that the tracking length, K, is short enough to assume that aircraft "i" is either completely present or completely absent over the entire K time segments. This assumption is not unrealistic.

The first stage decision is carried out by the ground processor using the space of vectors

$$\{[\rho(i,1),\ \rho(i,2),\ldots\rho(i,K)]\} \qquad (A.2)$$

as an observation space. A Neyman-Pearson Test, designed for a specific P_D, is set up using this observation space.

Let [r(i,1), r(i,2),...r(i,K)] be a specific realization of [ρ(i,1), ρ(i,2),...ρ(i,K)]. The Neyman-Pearson test utilizes the likelihood ratio Λ_1:

$$\Lambda_1 = \frac{\text{Prob } (\rho(i,1) = r(i,1),\ldots\rho(i,K) = r(i,K)|\ H_0(i))}{\text{Prob } (\rho(i,1) = r(i,1),\ldots\rho(i,K) = r(i,K)|\ H_1(i))} \tag{A.3}$$

Using this ratio the decision test is

$$\begin{aligned} &\text{decide } H_1(i) \text{ if } \Lambda_1 \le \lambda_1{}^K \\ &\text{decide } H_0(i) \text{ if } \Lambda_1 > \lambda_1{}^K \end{aligned} \tag{A.4}$$

and the following P_D is realized with the test

$$P_D(\lambda_1) = \text{Prob } (\Lambda_1 \le \lambda_1^K \mid H_1(i)) \tag{A.5}$$

Let K_0 equal the number of lists, in the set {List (i,j)}, which have no entries. The random variables r(i,j) are independent and identically distributed. The likelihood ratio, (A.3), may thus be equivalently written as

$$\Lambda_1 = \left(\frac{\text{Prob } (\rho(i,1) = 0|\ H_0(i))}{\text{Prob } (\rho(i,1) = 0|\ H_1(i)}\right)^{K_0} \left(\frac{\text{Prob } (\rho(i,1)= 1|\ H_0(i))}{\text{Prob } (\rho(i,1) = 1|\ H_1\ (i))}\right)^{K-K_0} \tag{A.6}$$

Using (A.6), the test (A.4) may be written as decide $H_1(i)$ if

$$K_o \left(\mathrm{Ln} \left(\frac{\mathrm{Prob}\ (\rho(i,1) = 0 \mid H_o(i))}{\mathrm{Prob}\ (\rho(i,1) = 0 \mid H_1(i))} \right) + \mathrm{Ln} \left(\frac{(\mathrm{Prob}\ (\rho(i,1) = 1 \mid H_1(i))}{\mathrm{Prob}\ (\rho(i,1) = 1 \mid H_o(i))} \right) \right)$$

$$\mathrm{Ln}\ \lambda_1 \qquad \text{(A.7)}$$

$$\leq K \left(-\mathrm{Ln} \left(\frac{\mathrm{Prob}\ (\rho(i,1) = 1 \mid H_o(i))}{\mathrm{Prob}\ (\rho(i,1) = 1 \mid H_1(i))} \right) \right) \qquad \text{(A.8)}$$

Decide $H_o(i)$ otherwise

The coefficient of the left hand side of (A.7) is positive. We may thus transpose this coefficient and observe that the decision test simplifies to a simple majority decision. Since the parameter λ_1 is at the disposal of the designer of the test, we may appropriately specify it so that our test can be simply written as

$$\text{Decide } H_1\ (i) \text{ if } K_o \leq t\ K$$

$$\text{Decide } H_o(i) \text{ if } K_o > t\ K \qquad \text{(A.9)}$$

where the test parameter t is an element of [0,1]. The detection probability which this test yields is

$$P_D = \mathrm{Prob}\ (K_o \leq t\ K \mid H_1(i)) \qquad \text{(A.10)}$$

We have suppressed the dependence on t. We shall now lower bound P_D.

First, consider the case when K equals 1. The only meaningful value for t in this case is zero. The decision test then becomes

$$\text{Decide } H_1(i) \text{ if } K_o = 0$$

$$\text{Decide } H_0(i) \text{ if } K_o = 1$$

and the detection probability can be lower bounded as

$$P_D \geq p_d^4 \tag{A.10a}$$

Now, consider the case when the tracking length K is strictly greater than 1. Observing (A.10), P_D may be rewritten as

$$
\begin{aligned}
P_D &= \text{Prob } (K - K_o > t K \mid H_1(i)) \\
&= \text{Prob } (-K_o > K(t-1) \mid H_1(i)) \\
&= \text{Prob } (K_o < K(1-t) \mid H_1(i)) \\
P_D &= 1 - \text{Prob } (K_o \geq K (1-t) \mid H_1(i))
\end{aligned} \tag{A.11}
$$

Using Chernoff bound we have

$$\text{Prob } (K_o \geq K (1-t) \mid H_1(i)) \leq e^{-\nu K(1-t)} \underset{\sim}{E}\, e^{\nu K_o} \tag{A.12}$$

The expectation is taken under the condition that $H_1(i)$ is true. We note that

$$K_o = \sum_{j=1}^{K} (1-\rho(i,j))$$

in which case (A.12) becomes

$$\text{Prob } (K_o \geq K(1-t) \mid H_1(i)) \leq \exp\left(-\nu K(1-t) + K\text{Ln } \underset{\sim}{E}\, e^{\nu(1-\rho(i,1)}\right) \tag{A.13}$$

(The notation exp (x) denotes e^{x})

The parameter ν in (A.13) is non-negative but otherwise arbitrary at this point.

Now,

$$\underset{\sim}{E}\, e^{\nu(1-\rho(i,1))} = P(\rho(i,1) = 0 |\, H_1(i))\, e^{\nu} + P(\rho(i,1) = 1 |\, H_1(i)) \quad (A.14)$$

Applying (A.14) to (A.13) and the result to (A.11) yields

$$P_D \geq 1 \exp -K\, (\nu(1-t) - Ln\, (P\, (\rho(i,1) = 0 |\, H_1(i))\, e^{\nu} + P\, (\rho(i,1) = 1 \,|\, H_1(i))) \quad (A.15)$$

Now

$$P(\rho(i,1) = 0 |\, H_1(i)) \simeq 1 - p_d^{\,4}$$

(A.16)

$$P(\rho(i,1) = 1 |\, H_1(i)) \simeq p_d^{\,4}$$

where the approximation holds closely assuming reasonably large enough values for p_d. Specifically, if $P(\rho(i,1) = 1 \mid H_1(i))$ is expanded into probabilities of the events component to $\{\rho(i,1) = 1 |\, H_1(i)\}$ then the term $p_d^{\,4}$ will dominate since all other terms will have a factor of $(1-p_d)$. When the approximation doesn't hold exactly, the lower bound of (A.15) will still be true if the approximation is used. Applying (A.16) to (A.15) yields

$$P_D \geq 1 - \exp -K\, (\nu(1-t) - Ln\, ((1-p_d^{\,4})e^{\nu} + p_d^{\,4})) \quad (A.17)$$

ν is now picked to maximize

$$\nu(1-t) - Ln\, ((1-p_d^{\,4})\, e^{\nu} + p_d^{\,4}) \quad (A.18)$$

One may note in observing (A.17) that if $t < p_d^4$ then P_D will approach 1 as K gets larger and larger. Similarly if P_D is fixed at some value and K is fixed at some value, then (A.17) can be made to hold by taking t sufficiently small and p_d sufficiently large. In any event (A.17) is the desired lower bound for K > 1 and

$$P_D > p_d^4$$

APPENDIX B

A LOWER BOUND TO P_F

We derive a lower bound to the parameter P_F in this appendix where we define (by Eq. 2.5.3)

$$P_F = P\ (\gamma_F(i) = 1 | \Theta_F(i)=1)$$

In the terminology of the first stage decision process described in Appendix A this can be rewritten as

$$P_F = \text{Prob}\ (K - K_0 > tK \mid H_0(i))$$

which becomes

$$P_F = \text{Prob}\ (K_0 < (1-t)\ K | \ H_0(i)) \qquad (B.1)$$

First, lets consider the case when K=1. **t** is now zero. P_F simplifies to

$$P_F = \text{Prob}\ (\rho\ (i,1) = 1 | \ H_0(i)) \qquad (B.2)$$

Consider the following event inclusion under the condition that $H_0(i)$ is true

$$\{\rho(i,1) = 1\} \supset \Phi_1 \qquad (B.3)$$

where Φ_1 is defined as

$$\Phi_1 = \left\{ \begin{array}{l} \text{consider Seg. (1) and the output of Filter (i,1) during} \\ \text{it. In the right hand subsegment of length } \alpha \text{ seconds, a} \\ \text{left most false declaration of the arrival of aircraft} \\ \text{"i's" signature at a sample point is declared by the} \\ \text{binary threshold decision process. Simultaneously, at} \\ \text{the outputs of Filter (i,2), Filter (i,3) and Filter (i,4)} \\ \text{left most false signature arrival times are declared in} \\ \text{subsegments centered at the point where the noted false} \\ \text{declaration in the subsegment of Filter (i,1)'s output} \\ \text{occurred, and having radii } \beta. \end{array} \right\} \quad \text{(B.4)}$$

where Filter (i,j) is the filter processing the output of the jth satellite matched to the ith signal. We assume in (B.4) that time in the matched filter outputs is proceeding from left to right. If event Φ_1 occurs, it will surely imply an entry on List (i,1) since it corresponds to a valid signature arrival time triplet. The condition in the event definition that the signatures arrive within β seconds of each other, (β is the maximum time delay between a signature reception from two different satellites), insures this. Thus, from (B.3) we have

$$\text{Prob } (\rho \text{ (i,1)} = 1 | H_0(i)) \geq \text{Prob } (\Phi_1 \quad H_0(i)) \quad \text{(B.5)}$$

In a subsegment of length α seconds there are $2B\alpha$ matched filter samples considered by the binary threshold decision process. In a subsegment of length 2β seconds there are $4B\beta$ matched filter samples considered we have then

$$\text{Prob}(\Phi_1 \mid H_0(i)) = \left(1-(1-p_f)^{2B\alpha}\right)\left(1-(1-p_f)^{4B\beta}\right)^3 \tag{B.6}$$

and

$$\text{Prob}(\rho(i,1) = 1 \mid H_0(i)) \geq \left(1-(1-p_f)^{2B\alpha}\right)\left(1-(1-p_f)^{4B\beta}\right)^3 \tag{B.7}$$

(B.7) can be applied to (B.2) to obtain

$$P_F \geq \left(1-(1-p_f)^{2B\alpha}\right)\left(1-(1-p_f)^{4B\beta}\right)^3 \tag{B.8}$$

when K=1.

We now consider the case when K is greater than one. The following event inclusion, under the restriction that $H_0(i)$ is true, should be obvious

$$\{\text{ A false alarm on Seg.(K)}\} \supseteq \bigcap_{j=1}^{Kt+1} \{\rho(i,j) = 1\} \tag{B.9}$$

From (B.8) and the statistical independence of $\rho(i,j)$, we obtain immediately

$$P_F \geq (\text{Prob}(\rho(i,j) = 1 \mid H_0(i)))^{Kt+1} \tag{B.10}$$

Applying (B.5) and (B.6) to (B.10) results in

$$P_F \geq \exp\left[(Kt+1)\,\text{Ln}\left(\left(1-(1-p_f)^{2B\alpha}\right)\left(1-(1-p_f)^{4B\beta}\right)^3\right)\right] \tag{B.11}$$

Which is the desired lower bound and which gives the lower bound for the case when K=1 under the convention that t=0 in this case.

APPENDIX C

A LOWER BOUND TO PROB $[\gamma_D(i)=1]$

In this Appendix, we derive a lower bound to Prob $[\gamma_D(i) = 1]$. As in previous Appendices, we first consider the case when K=1.

When K=1 the event $[\gamma_D(i) = 1]$ includes the event Φ_1, defined by (B.4). Hence,

$$[\gamma_D(i) = 1] \supseteq \varphi_1$$

We obtain then directly from (B.6)

$$\text{Prob } [\gamma_D(i) = 1] \geq [1 - p_f)^{2B\alpha}] \, [1 - (1-p_f)^{4B\beta}]^3 \tag{C.1}$$

This is the desired lower bound when K=1.

We can now concentrate our efforts on the case when K is greater than 1. Our derivation is begun by considering the definition of $\gamma_D(i)$ as defined in subsection 2.5.2. We will be more explicit now about what we mean by saying that a sequence of time difference triplets appears as if it were generated by an aircraft in flight.

Consider aircraft "i" in flight and two successive transmissions of aircraft "i's" signature. Let us call the time difference triplets; (d_1^1, d_2^1, d_3^1) and (d_1^2, d_2^2, d_3^2). For our purposes it will be enough just to deal with d_1^1 and d_1^2. Now,

$$d_1^1 = T_1(1) - T_1(2)$$

$$d_1^2 = T_2(1) - T_2(2)$$

where

$$T_1\ (1) = \left\{\begin{array}{l}\text{Signature reception time of aircraft "i's" true}\\ \text{signature from Satellite 1 during the earlier of the}\\ \text{sucessive transmissions.}\end{array}\right\}$$

$$T_1\ (2) = \left\{\begin{array}{l}\text{Signature reception time of aircraft "i's" true}\\ \text{signature from Satellite 2 during the earlier of the}\\ \text{sucessive transmissions.}\end{array}\right\}$$

$$T_2\ (1) = \left\{\begin{array}{l}\text{Signature reception time of aircraft "i's" true}\\ \text{signature from Satellite 1 during the later of the}\\ \text{successive transmissions.}\end{array}\right\}$$

$$T_2\ (2) = \left\{\begin{array}{l}\text{Signature reception time of aircraft "i's" true}\\ \text{signature from Satellite 2 during the later of the}\\ \text{sucessive transmissions.}\end{array}\right\}$$

Let us look at what maximum value $\left|d_1^1 - d_1^2\right|$ can have

$$|d_1^1 - d_1^2| = |T_1(1) - T_1(2) - T_2(1) + T_2(2)|$$

$$|d_1^1 - d_1^2| = |(T_1(1) - T_2(1)) - (T_1(2) - T_2(2))| \qquad (C.2)$$

$$|d_1^1 - d_1^2| \leq |T_1(1) - T_2(1)| + |T_1(2) - T_2(2)|$$

Let

$$v = \text{maximum aircraft velocity} \qquad (C.3)$$

$$c = \text{speed of light} \qquad (C.4)$$

where v and c are expressed in the same units of miles per second. The maximum radial distance that any aircraft can move from a satellite between two successive transmissions is $v\alpha$ miles. Therefore,

$$|T_1(1) - T_2(1)| \leq \frac{v\alpha}{c} \text{ seconds}$$

$$|T_1(2) - T_2(2)| \leq \frac{v\alpha}{c} \text{ seconds}$$

Combining these inequalities with (C.2) we conclude that

$$|d_1^1 - d_1^2| \leq \frac{2v\alpha}{c} \qquad (C.5)$$

(C.2) can now be used to define what we mean by two time difference triplets satisfying the constraints of aircraft flight. Because aircraft, particularly the great majority of general aviation aircraft rarely fly at the maximum velocity, v, we shall be more strigent than (C.4) in defining the constraints:

Definition C.1: Let (d_1^1, d_2^1, d_3^1) be a time difference triplet on List (i,m) and let (d_1^2, d_2^2, d_3^2) be a time difference triplet on List (i,m+1), m=1,...K-1. These two time difference triplets satisfy the constraints of aircraft flight if

$$|d_1^1 - d_1^2|,\ |d_2^1 - d_2^2|,\ |d_3^1 - d_3^2| \leq \frac{2}{3}\ \frac{v\alpha}{c} \qquad \text{(C.6)}$$

With this definition we can reconsider the definition of the indicator function, $\gamma_D(i)$ as defined in subsection 2.5.2. With respect to subsection 2.5.2, we mean by the statement: "This sequence of time difference triplets appears as if it were generated by an aircraft in flight." That any two successive time difference triplets in the sequence satisfy the constraints of aircraft flight as defined by Definition C.1. For convenience, we shall now introduce the indicator function $\gamma_D(i,m)$, which is similar to $\gamma_D(i)$, the event $\delta(m+1)$, and the term Δ.

$\gamma_D(i,m) = 1$, if there is a first sequence of m arrival time difference triplets, (T(1)-T(2), T(1)-T(3), T(1)-T(4)), with the first sequence member on List (i,1),... the mth sequence member on List (i,m). The elements of this sequence have all their component arrival times, T( ), only generated by interference (not interference plus aircraft "i's" signature) received at [Filter (i,j)]. This sequence of time difference triplets appears as if it were generated by an aircraft in flight.

(C.7)

$= 0$, otherwise

$m = 1,\ldots K-1$

$$\delta(m+1) = \left\{ \begin{array}{l} \text{A time difference triplet is completely generated by inter-} \\ \text{ference on List (i, m+1) and this time difference triplet} \\ \text{satisfies the constraints of aircraft flight with the time} \\ \text{difference triplet on List (i,m) which belongs to the first} \\ \text{sequence causing } \gamma_D(i,m) = 1. \end{array} \right\}$$

$$m = 1, 2, \ldots K - 1 \tag{C.8}$$

$$\Delta = \left\{ \begin{array}{l} \text{The time difference triplet on List (i,m) corresponding} \\ \text{to the first sequence causing } \gamma_D(i,m) = 1. \text{ Call the} \\ \text{components of } \Delta, (d_1, d_2, d_3) \end{array} \right\} \tag{C.9}$$

Obviously,

$$[\gamma_D(i) = 1] = [\gamma_D(i, K - 1] \cap \delta(K) \tag{C.10}$$

in which case

$$\text{Prob } [\gamma_D(i) = 1] = \text{Prob } [\delta(k) | \gamma_D(i,K) = 1] \text{ Prob } [\gamma_D(i, K - 1) = 1] \tag{C.11}$$

$$\text{Prob } [\gamma_D(i) = 1] = \sum_{\Delta} \text{Prob} \left\{ \delta(K) \middle| \begin{array}{l} [\gamma_D(i,K-1)=1] \cap \\ \Delta = (d_1, d_2, d_3) \end{array} \right\} \text{Prob}[\Delta = (d_1 d_2 d_3) | \gamma_D(i,K-1)=1]$$

$$\text{Prob}[\gamma_D(i,K-1) = 1] \tag{C.12}$$

The following event inclusion should be obvious under the condition that $[\gamma_D(i,K-1)=1] \cap [\Delta=(d_1,d_2,d_3)]$ has occurred.

$$\delta(K) \supseteq \left\{ \begin{array}{l} \text{Consider Seg. (K) and the output of Filter (i,1) during it.} \\ \text{In the right hand subsegment of length } \alpha \text{ seconds, a left} \\ \text{most false declaration of the arrival of aircraft "i's"} \\ \text{signature at a sample point is declared by the binary} \\ \text{threshold decision process. Simultaneously, at the outputs} \\ \text{of Filters (i,j), j=2, 3, 4, left most false signature} \\ \text{arrival times are declared in subsegment centered at a} \\ \text{distance } d_{j-1} \text{ from the point where the noted false declara-} \\ \text{tion in the subsegment of Filter (i,1)'s output occurred, and} \\ \text{having radius } \frac{2}{3}\frac{v\alpha}{c} \text{ .} \end{array} \right. \tag{C.13}$$

In (C.13) we mean by a "false declaration" a declaration generated only by interference, (C.13) immediately gives

$$\text{Prob}\left\{\delta(K) \,\middle|\, \begin{array}{l}[\gamma_D(i,K-1)=1] \cap \\ \Delta=(d_1,d_2,d_3)\end{array}\right\} \geq [1-(1-p_f)^{2B\alpha}] \left[1-(1-p_f)^{\frac{8B}{3}\frac{v\alpha}{c}}\right]^3 \tag{C.14}$$

Applying (C.14) to (C.12) we obtain

$$\text{Prob}[\gamma_D(i)=1] \geq [1-(1-p_f)^{2B\alpha}] \left[1-(1-p_f)^{\frac{8B}{3}\frac{v\alpha}{c}}\right]^3 \text{Prob}[\gamma_D(i,K-1)=1] \tag{C.15}$$

The event $[\gamma_D(i,K-1)=1]$ can be decomposed into $[\gamma_D(i,K-2)=1] \cap \delta(K-1)$ just as $[\gamma_D(i)=1]$ was similarly decomposed into $[\gamma_D(i,K-1)=1] \cap \delta(K)$ in (C.8). $\text{Prob}[\gamma_D(i,K-1)=1]$ can be lower bounded as $\text{Prob}[\gamma_D(i)=1]$ was in (C.15). The result can be applied to (C.15). The same procedure can be applied successively to the events; $[\gamma_D(i, K-2)=1],\ldots[\gamma_D(1,2)=1]$, yielding

$$\text{Prob}[\gamma_D(i)=1] \geq \left\{[1-(1-p_f)^{2B\alpha}]\left[1-(1-p_f)^{\frac{8B}{3}\frac{v\alpha}{c}}\right]^3\right\}^{K-1} \text{Prob}[\gamma_D(i,1)=1] \tag{C.16}$$

$\text{Prob}\,[\gamma_D(i,1)=1]$ is the same as the $\text{Prob}\,[\gamma_D(i)=1]$ when $K=1$. This is obtained from (C.1). When this is applied to (C.14), we have as a result

$$\text{Prob}[\gamma_D(i)=1] \geq \left\{[1-(1-p_f)^{2B\alpha}]\left[1-(1-p_f)^{\frac{8B}{3}\frac{v\alpha}{c}}\right]^3\right\}^{K-1}[1-(1-p_f)^{2B\alpha}]$$

$$[1-(1-p_f)^{4B\beta}]^3 \tag{C.17}$$

$$\text{Prof}[\gamma_D(i)=1] \geq \exp\left\{(K-1)\text{Ln}[1-(1-p_f)^{2B\alpha}]\left[1-(1-p_f)^{\frac{8B}{3}\frac{v\alpha}{c}}\right]^3\right.$$

$$\left. + \text{Ln}[1-(1-p_f)^{2B\alpha}]\,[1-(1-p_f)^{4B\beta}]^3\right\}$$

which, when we compare to (C.1), we see is the desired lower bound when either $K=1$ or $K>1$.

APPENDIX D

THE DEPENDENCE OF p_f ON SYSTEM AND JAMMER PARAMETERS

In this Appendix we shall derive an expression for p_f in terms of both the system and jammer parameters.

Consider one of the matched filters at the ground station matched to aircraft "i's" signature. Let "y" be one of the matched filter output samples. We define the following hypotheses:

$$h_o^* = \{y \text{ is generated by noise along}\} \quad (D.1)$$

$$h_1^* = \{y \text{ is generated by a combination of noise and aircraft "i's" signature}\} \quad (D.2)$$

We assume that the binary threshold decision process sets up a Neyman-Pearson test on the sample to guarantee a "sample detection probability" of at least p_d. The threshold, λ, used for the test is the solution of

$$\text{Prob } [\Lambda(y) \leq \lambda | h_1^*] = p_d \quad (D.3)$$

where

$$\Lambda(y) = \frac{P_{y|h_o^*}(y|h_o^*)}{P_{y|h_1^*}(y|h_1^*)} \quad (D.4)$$

($P_{y|h_0^*}$ and $P_{y|h_1^*}$ are the conditional probability densities of y)

The Neyman-Pearson test is:

$$\text{decide } h_1^* \text{ if } \Lambda(y) \leq \lambda$$
$$\text{decide } h_0^* \text{ if } \Lambda(y) > \lambda \qquad (D.5)$$

and it operates with

$$p_f = \text{Prob}\,[\Lambda(y) \leq \lambda | h_0^*] \qquad (D.6)$$

We assume that if the signature, $s_i(t)$, contributes to the matched filter output sample, it contributes the maximum amplitude that it can, i.e. $\sqrt{E}$.

The white gaussian noise transmitted at a satellite by the jammer is relayed to the ground station. It appears as white gaussian noise with power density height J/B at the input to the matched filter. This appears as gaussian at the output of the matched filter with a noise power of J/B watts. From this and the preceeding paragraph we immediately obtain the probability density of output y when h_1^* is true:

$$f(y|h_1^*) = \frac{1}{\sqrt{2\pi J/B}} \exp\left[\frac{-(y - \sqrt{E})^2}{2J/B}\right] \qquad (D.7)$$

and the probability density when h_o^* is true

$$f(y|h_o^*) = \frac{1}{\sqrt{2\pi J/B}} \exp\left(\frac{-y^2}{2J/B}\right) \tag{D.8}$$

Applying (D.7) and (D.8) to (D.4) and the result to (D.6) yields

$$p_f = \frac{1}{\sqrt{2\pi J/B}} \int_{\frac{\sqrt{E}}{2} - \frac{J/B\ Ln\lambda}{\sqrt{E}}}^{\infty} \exp\left(\frac{-y}{2J/B}\right) dy$$

which can be rewritten as

$$p_f = \frac{1}{\sqrt{\pi}} \int_{\frac{1}{2}\sqrt{\frac{E}{2J/B}} - \sqrt{\frac{J/B}{2E}}\ Ln\lambda}^{\infty} \exp(-y^2) dy$$

and which can be further simplified to

$$p_f = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{E}{2J/B}} - \sqrt{\frac{J/B}{2E}}\ Ln\lambda\right) \tag{D.9}$$

In terms of (D.4), p_d can be expressed as

$$p_d = \text{Prob}[\Lambda(y) \leq \lambda | h_1^*] \tag{D.10}$$

applying (D.7) and (D.8) to D.4) and the result to (D.10) yields

$$p_d = \frac{1}{\sqrt{2\pi J/B}} \int_{\frac{\sqrt{E}}{2} - \frac{J/B\ Ln\lambda}{\sqrt{E}}}^{\infty} \exp\left[\frac{-(y-\sqrt{E})^2}{2J/B}\right] dy \tag{D.11}$$

(D.11) can be rewritten as

$$p_d = \frac{1}{\sqrt{\pi}} \int_{\frac{1}{2}\sqrt{\frac{E}{2J/B}} - \sqrt{\frac{J/B}{2E}}\ Ln\lambda}^{\infty} \exp(-y^2)\ dy \tag{D.12}$$

which may be further simplified to

$$p_d = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{E}{2J/B}} + \sqrt{\frac{J/B}{2E}}\ Ln\ \lambda\right) \tag{D.13}$$

Consider the fixed value of p_d and let ϕ be the solution of

$$p_d = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\ (\phi) \tag{D.14}$$

We must have the following equality satisfied

$$\phi = \frac{1}{2}\sqrt{\frac{E}{2J/B}} + \sqrt{\frac{J/B}{2E}}\ Ln\ \lambda \tag{D.15}$$

Applying (D.15) to (D.9) results in

$$p_f = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\sqrt{\frac{E}{2J/B}} - \phi\right) \tag{D.16}$$

Note that $E = P_o \tau$, where P_o is peak signature power. We then obtain

$$p_f = \frac{1}{2} - \frac{1}{2} \operatorname{erf}\left(\sqrt{\frac{1}{2} \frac{P_o}{J} B \tau} - \phi\right) \qquad \text{(D.17)}$$

where ϕ is the solution of

$$p_d = \frac{1}{2} + \frac{1}{2} \operatorname{erf}(\phi)$$

(D.17) gives the desired dependence of p_f upon the parameters of interest.

APPENDIX E

PROOF OF THEOREM 3.1

We begin by applying the restriction on a common amplitude density to (3.4.6). This yields

$$M_i(v) = \exp\left((N-1)\, Ln\left(\left(1-\frac{2\tau}{\alpha}\right) + \frac{2\tau}{\alpha}\int_{-\infty}^{\infty} C(x)\, e^{jvx}\, dx\right)\right) \qquad (E.1)$$

Now we just need to analyze the asymptotic behavior of this expression for $M_i(v)$. Because we are considering asymptotic behavior with N, we can replace the coefficient (N-1) with N, without any error in our work. In order to investigate the asymptotic behavior of $M_i(v)$ here, we shall use the same procedures utilized by Rice in his investigation of shot noise [2].

It will be convenient to define the following density function

$$Q(x) = \left(1-\frac{2\tau}{\alpha}\right) u_0(x) + \frac{2\tau}{\alpha} C(x) \qquad (E.2)$$

Applying (E.2) to (E.1) yields

$$M_i(v) = \exp\left(N\, Ln\left(\int_{-\infty}^{\infty} Q(x)\, e^{jvx}\, dx\right)\right) \qquad (E.3)$$

Now,

$$Ln\left(\int_{-\infty}^{\infty} Q(x) e^{jvx} dx\right) = \sum_{r=1}^{\infty} \frac{\lambda_r}{r!} (jv)^r \qquad (E.4)$$

where λ_r is the r^{th} semi-invariant of density Q(x), i.e.,

$$\lambda_r = \frac{1}{j^r} \frac{\partial^r}{\partial v^r} \, Ln \left(\int_{-\infty}^{\infty} Q(x) \, e^{jvx} dx \right) \Bigg|_{v=0} \qquad (E.5)$$

(E.4) is merely the Taylor expansion of $Ln(\int_{-\infty}^{\infty} Q(x) \, e^{jvx} dx)$ around v=0.

When (E.4) is applied to (E.3) the result is

$$M_i(v) = \exp \left(N \sum_{r=1}^{\infty} \frac{\lambda_r}{r!} (jv)^r \right) \qquad (E.6)$$

Let G(n) be the probability density on $n_i(t)$. Using Fourier inversion (E.6) yields

$$G(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[-jnv + N \sum_{r=1}^{\infty} \frac{\lambda_r}{r!} (jv)^r \right] dv \qquad (E.7)$$

Applying (3.4.9) to (E.5) one obtains

$$\lambda_1 = 0 \qquad (E.8)$$

$$\lambda_2 = \frac{2\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x) \, dx \qquad (E.9)$$

Substituting (E.8) and (E.9) into (E.7) brings

$$G(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-jnv - \left(\frac{2N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x) \, dx\right) \frac{v^2}{2}\right] \exp\left[N \sum_{r=3}^{\infty} \frac{\lambda_r}{r!} (jv)^r\right] dv \qquad (E.10)$$

We can now expand $\exp [N \sum_{r=3}^{\infty} \frac{\lambda_r}{r!} (jv)^r]$ in a power series in its exponent obtaining

$$\exp [N \sum_{r=3}^{\infty} \frac{\lambda_r}{r!} (jv)^r] = \sum_{m=0}^{\infty} \frac{1}{m!} [N \sum_{r=3}^{\infty} \frac{\lambda_r}{r!} (jv)^r]^m \quad (E.11)$$

Collecting like powers of v (E.11) becomes

$$\exp [N \sum_{r=3}^{\infty} \frac{\lambda_r}{r!} (jv)^r] = 1 + \frac{N\lambda_3}{3!} (jv)^3 + \frac{N\lambda_4}{4!} (jv)^4 + \frac{N\lambda_5}{5!} (jv)^5 + [\frac{N\lambda_6}{6!} + \frac{1}{2} N^2 \frac{\lambda_3^2}{(3!)^2}] (jv)^6 + \ldots \quad (E.12)$$

Applying (E.12) to (E.10) and integrating term by term brings

$$G(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp [-jnv - (\frac{2N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx) \frac{v^2}{2}] \, dv$$

$$+ \frac{N\lambda_3}{3!} \frac{1}{2\pi} \int_{-\infty}^{\infty} (jv)^3 \exp [-jnv - (\frac{2N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx) \frac{v^2}{2}] dv$$

$$+ \frac{N\lambda_4}{4!} \frac{1}{2\pi} \int_{-\infty}^{\infty} (jv)^4 \exp [-jnv - (\frac{2N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx) \frac{v^2}{2}] dv$$

$$+ \frac{N\lambda_5}{5!} \frac{1}{2\pi} \int_{-\infty}^{\infty} (jv)^5 \exp [-jnv - (\frac{2N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx) \frac{v^2}{2}] dv$$

$$[+ \frac{N\lambda_6}{6!} + \frac{1}{2} \frac{N^2\lambda_3^2}{(3!)^2} \int_{-\infty}^{\infty}](jv)^6 \exp [-jnv - (\frac{2N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx) \frac{v^2}{2}] \, dv$$

$$+ \ldots \tag{E.13}$$

Let us abbreviate the Normal distribution on a random variable, "n", in the usual way as

$$\mathcal{N}(a,b) = \frac{1}{\sqrt{2\pi b}} \exp (- \frac{(n-a)^2}{2b}) \tag{E.14}$$

Recognizing the moment generating function of a normal distribution in (E.13) we may rewrite (E.13) as

$$G(n) = \mathcal{N}(0, \frac{2N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx)$$

$$- \frac{N\lambda_3}{3!} \frac{d^3}{dn^3} \mathcal{N}(0, \frac{2N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx) \tag{E.15}$$

$$+ \frac{N\lambda_4}{4!} \frac{d^4}{dn^4} \mathcal{N}(0, \frac{2N\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx) + \ldots$$

The first term in (E.15) is $O(N^{-1/2})$. The second term is $O(N^{-3/2})$. Higher order terms are also $O(N^{-3/2})$. Thus, for a fixed τ/α as N gets large G(n) is asymptotic to the first term and we have

$$G(n) \sim \mathcal{N}(0, \frac{N2\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C(x)dx)$$

which proves the theorem since $G(n)$ is the probability density on $n_i(t)$, $M_i(v)$ is the characteristic function corresponding to it.

APPENDIX F

PROOF OF THEOREM 3.3

We begin from (3.4.10)

$$E(n^2) = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{2\tau}{\alpha} \int_{-\infty}^{\infty} x^2 C_{ij}(x)dx \qquad (3.4.10)$$

By the definition of $C_{ij}(\cdot)$ given in Section 3.3 we have

$$\int_{-\infty}^{\infty} x^2 C_{ij}(x)dx = E\ (n_{ij}^2(t)) \qquad (F.1)$$

Applying the definition of $n_{ij}(t)$, (see (3.3.7)) to (F.1) yields

$$\int_{-\infty}^{\infty} x^2 C_{ij}(x)dx = \frac{1}{2}\ E\ [\int_{-\infty}^{\infty} S_j(x-t_j)\ S_i(x-t+\tau)dx]^2$$

The expectation is taken under the condition $t - 2\tau \leq t_j \leq t$

$$\int_{\infty}^{\infty} x^2 C_{ij}(x)dx = \frac{1}{2}\ E\ [\int_{\infty}^{\infty}\int_{-\infty}^{\infty} S_j(x-t_j)S_i(x-t+\tau)S_j(u-t_j)S_i(u-t+\tau)\ dx\ du] \qquad (F.2)$$

t_j will be uniformly distributed over $[t - 2\tau, t]$ hence (F.2) becomes

$$\int_{-\infty}^{\infty} x^2 C_{ij}(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{2\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_j(x-a)S_j(u-a)S_i(x-t+\tau)S_i(u-t+\tau)dx\ du\ da$$

We lose nothing by increasing the limits on "a" to $[-\infty,\infty]$. Doing this and reordering the integrals we have

$$\int_{-\infty}^{\infty} x^2 C_{ij}(x)dx = \frac{1}{4\tau} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_i(x-t+\tau)\ S_i(u-t+\tau)\ dxdu \int_{-\infty}^{\infty} S_j(x-a)S_j(u-a)da \quad (F.3)$$

Referring to the definition of $R_{jj}(t)$ given in the theorem statement, (F.3) can be rewritten as

$$\int_{-\infty}^{\infty} x^2 C_{ij}(x)dx = \frac{1}{4\tau} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_i(x-t+\tau)\ S_i(u-t+\tau)\ R_{jj}(x-u)\ dx\ du \qquad (F.4)$$

Let us make the change of variables $q = x-u$. Applying this to (F.4) yields

$$\int_{-\infty}^{\infty} x^2 C_{ij}(x)dx = \frac{1}{4\tau} \int_{-\infty}^{\infty} R_{jj}(q)\ dq \int_{-\infty}^{\infty} S_i\ (q+u-t+\tau)\ S_i(u-t+\tau)\ du$$

Again referring to the definition of $R_{jj}(t)$ this becomes

$$\int_{-\infty}^{\infty} x^2 C_{ij}(x)dx = \frac{1}{4\tau} \int_{-\infty}^{\infty} R_{jj}(q)\ R_{ii}(q)\ dq \qquad (F.5)$$

Applying (F.5) to (3.4.10) brings

$$E(n^2) = \frac{1}{2\alpha} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_{-\infty}^{\infty} R_{jj}(q)\ R_{ii}(q)\ dq \qquad (F.6)$$

By the Schwartz Inequality

$$\int_{-\infty}^{\infty} R_{jj}(q)\, R_{ii}(q)\, dq \leq \left(\int_{-\infty}^{\infty} R_{jj}^2\,(q)\, dq\right)^{1/2} \left(\int_{-\infty}^{\infty} R_{ii}^2\,(q)\, dq\right)^{1/2} \qquad \text{(F.7)}$$

with equality iff $R_{ii}(\cdot) = R_{jj}(\cdot)$.

Substituting (F.7) into (F.6) brings

$$E(n^2) \leq \frac{1}{2\alpha} \sum_{\substack{j=1 \\ j\neq i}}^{N} \left(\left(\int_{-\infty}^{\infty} R_{jj}^2\,(q)\, dq \right)^{-1} \right)^{-1/2} \left(\left(\int_{-\infty}^{\infty} R_{ii}^2\,(q) dq \right)^{-1} \right)^{-1/2} \qquad \text{(F.8)}$$

When the definition of Zakai bandwidth, given in the theorem, is applied to (F.8) we have

$$E(n^2) \leq \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{4\alpha \sqrt{W_i W_j}}$$

with equality if $R_{ii}(\cdot) = R_{jj}(\cdot)$ for all $j \neq i$. This proves the theorem.

APPENDIX G

A STOCHASTIC MODEL FOR THE SEQUENCE OF TRANSMISSION TIMES $\{\tau_n\}$

The sequence $\{\tau_n\}$ represents the sequence of times at which the aircraft transmits pulses. It is generated by the aircraft clock. Random distrubances in this clock cause $\{\tau_n\}$ to be a random sequence. As described in Section 4, in some instances we would like to track, i.e., casually estimate, the sequence $\{\tau_n\}$; and in other instances, predict the values of $\{\tau_n, n\geq K+1\}$ after having tracked the subsequence $\{\tau_n, n\leq K\}$. In order to determine the structures of these tracking and predicting systems, and determine the resulting errors, we need a reasonably accurate stochastic model for $\{\tau_n\}$.

The heart of the clock which generates $\{\tau_n\}$ is an oscillator with some nominal frequency f_o. In addition to the oscillator, the clock consists of a counting mechanism, in the form of a digital circuit, which counts completed cycles in the oscillator output. When the counter has registered N cycles, the aircraft transmits a pulse and the counter is set to zero. Therefore, the sequence $\{\tau_n\}$ is defined as

$$\tau_n = t_{K(nN)} + \gamma_n$$

where t_K is the time of completion of the kth cycle, K(nN) is a random variable equal to the number of cycles that must be completed before nN cycles are <u>detected</u> by the counter, and γ_n is the error in detecting the exact completion time of the K(nN)th cycle.

In an effort to simplify the model for $\{\tau_n\}$, while at the same time emphasizing the unusual and important aspects of the model, we shall assume that

$$K(nN) = nN$$

$$\gamma_n = 0$$

To justify these simplications, we shall shortly see that t_k has a component for which the variance increases as $k^{2\rho}$, where ρ is some number slightly less than one. Since, for $k << nN$,

$$(nN + K)^{2\rho} - (nN)^{2\rho} \cong 2\rho(nN)^{2\rho - 1}k$$

we can approximately model the randomness of $K(nN)$ by including in t_k a component for which the variance increases as $k^{2\rho - 1}$, or simply as k, since $\rho \simeq 1$. This will correspond to an independent increment component in t_k. Finally, as the variance in t_k will be increasing rapidly, we can assume that a non-zero γ_n contributes only slightly to the randomness of τ_n.

Having simplified the model for $\{\tau_n\}$ to

$$\tau_n = t_{nN}$$

we need a model for the sequence $\{t_k\}$. If $f(t)$ denotes the instantaneous frequency of the oscillator, then t_k is just the time when the accumulated phase equals $2\pi k$, i.e.

$$\int_0^{t_k} 2\pi f(t)dt = 2\pi k$$

The instantaneous frequency f(t) is a random process that may be decomposed as

$$f(t) = f_0 + f_\varepsilon(t)$$

where f_0 is the nominal frequency, i.e. a constant, and $f_\varepsilon(t)$ is the random component. We shall assume that there exists a time T, on the order of at least several minutes, and a number λ, on the order of 10^{-9}, such that

$$|f_\varepsilon(t)|<\lambda f_0, \text{ for all } 0\leq t\leq T \qquad (G.1)$$

We now need an expression for t_k where

$$\int_0^{t_k} [f_0 + f_\varepsilon(t)]dt = k$$

Defining the function G(t) as

$$G(t) = \int_0^t [f_0 + f_\varepsilon(s)]ds - k$$

we see that we are looking for the zero of G(t). This root may be found by a gradient search, which yields the sequence:

$$t_k^{(0)} = \frac{k}{f_0}$$

$$t_k^{(i)} = t_k^{(i-1)} - \frac{G(t_k^{(i-1)})}{G(t_k^{(i-1)})}$$

$$= t_k^{(i-1)} - \frac{\int_0^{t_k^{(i-1)}} [f_0 + f_\varepsilon(s)]ds - k}{f_0 + f_\varepsilon(t_k^{(i-1)})}$$

Using (G.1), we get the following sequence:

$$t_k^{(0)} = \frac{k}{f_0}$$

$$t_k^{(1)} = \frac{k}{f_0}\left[1 - \frac{f_0}{k}\int_0^{\frac{k}{f_0}} \frac{f_\varepsilon(t)}{f_0}\, dt\right]$$

$$\vdots$$

Again using (G.1) it is easily shown that this sequence converges to t_k, and that the error in $t_k^{(1)}$ is only on the order of $\frac{k}{f_0}\ \lambda^2$. Thus, to a very good approximation,

$$t_k = \frac{k}{f_0} - \frac{1}{f_0}\int_0^{\frac{k}{f_0}} f_\varepsilon(t)dt$$

If the correlation function for $f_\varepsilon(t)$ were concentrated near the time origin, then, for values of k/f_0 larger than the correlation time of $f_\varepsilon(t)$, the variance of $\int_0^{k/f_0} f_\varepsilon(t)dt$ would increase linearly with k, and we could effectively model $f_\varepsilon(t)$ as being white. However, measurements on oscillators have shown that, for values of k/f_0 greater than about one second, the variance of $\int_0^{k/f_0} f_\varepsilon(t)dt$ increases as $k^{2\rho}$, for some ρ just slightly smaller than one. This behavior of the variance has been attributed to "flicker" or "1/f" noise[5].

We need to develop a model for the random process

$$x(t) = -\int_0^t \frac{f_\varepsilon(s)}{f_0}\, ds$$

One model which matches most of the phenomena observed in oscillators results from decomposing x(t) as

$$x(t) = x_1(t) + x_2(t) + x_3(t)$$

where $x_1(t)$, $x_2(t)$, and $x_3(t)$ are zero mean, uncorrelated random processes. Further,

$$x_1(t) = \int_0^t b\,ds = bt$$

$$x_2(t) = \alpha\int_0^t w(s)ds$$

$$x_3(t) = \frac{\beta}{\Gamma(\rho + \frac{1}{2})} \left\{ \int_{-\infty}^{0} [(t - s)^{\rho - \frac{1}{2}} - (-s)^{\rho - \frac{1}{2}}] v(s) ds + \int_{0}^{t} (t - s)^{\rho - \frac{1}{2}} v(s) ds \right\}$$

where b is a random variable, w(s) and v(s) are uncorrelated white noise processes with unit spectral height, α and β are normalization constants, and ρ is a number satisfying

$$\frac{1}{2} < \rho < 1$$

The process $x_1(t)$ models very slowly varying offsets in the frequency; $x_2(t)$ models an independent increment component of x(t); and $x_3(t)$, when ρ is very close to one, models the "1/f" noise. The model for $x_3(t)$ is due to Mandelbrot, and we shall use several results, without proof, from [6]. We may interpret $x_3(t)$ as the $(\rho + \frac{1}{2})$ - fold integral of white noise (Note that when $\rho = \frac{1}{2}$, $x_3(t)$ is just the integral of v(t). As a result, the power density spectrum $S_{x_3}(f)$, if it existed, would behave as $|f|^{-2\rho-1}$, which is what the spectrum of the integral of "1/f" noise would be, if it existed.

In determining the second order statistics for x(t), we have by assumption

$$E[x(t)] = 0$$

and

$$\mathrm{Var}[x(t)] = \mathrm{Var}[x_1(t)] + \mathrm{Var}[x_2(t)] + \mathrm{Var}[x_3(t)]$$

Also,

$$\mathrm{Var}[x_1(t)] = \sigma_b^2 t^2$$

$$\mathrm{Var}[x_2(t)] = \alpha^2 t$$

and, from Ref. [6],

$$\mathrm{Var}[x_3(t)] = \beta^2 V_o t^{2\rho}$$

where

$$V_o = \frac{\frac{1}{2\rho} + \int_0^\infty [s^\rho - \frac{1}{2} - (s + 1)^\rho - \frac{1}{2}]^2 ds}{[\Gamma(\rho + \frac{1}{2})]^2}$$

For the correlation functions, we have

$$R_{x_1}(t, u) = \sigma_b^2 tu$$

$$R_{x_2}(t, u) = \alpha^2 \min(t, u)$$

To find $R_{x_3}(t, u)$, we use the fact that $x_3(t)$ has stationary increments, i.e. that

$$\mathrm{Var}[x_3(t) - x_3(u)] = \mathrm{Var}[x_3(t - u)], \; t>u$$

Then, since

$$\mathrm{Var}[x_3(t) - x_3(u)] = \mathrm{Var}[x_3(t)] + \mathrm{Var}[x_3(u)] - 2R_{x_3}(t, u)$$

we have

$$R_{x_3}(t, u) = \frac{1}{2}\left\{\mathrm{Var}[x_3(t)] + \mathrm{Var}[x_3(u)] - \mathrm{Var}[x_3(t) - x_3(u)]\right\}$$

$$= \frac{\beta^2 V_o}{2}\left(t^{2\rho} + u^{2\rho} - |t - u|^{2\rho}\right)$$

Combining the above, we now have

$$E[x(t)] = 0$$

$$R_x(t, u) = \sigma_b^2 tu + \alpha^2 \min(t, u)$$

$$+ \frac{\beta^2 V_o}{2}\left(t^{2\rho} + u^{2\rho} - |t - u|^{2\rho}\right)$$

It is apparent that, when $\rho \approx 1$, the effects of the "1/f" noise are quite similar to those of the unknown frequency offset, b. Unless σ_b^2 is quite large, any efforts at estimating b will be thwarted by the presence of the "1/f" noise. Therefore, to further simplify the model, we now assume

$$\sigma_b^2 = 0$$

We have reduced the model for x(t) so that

$$E[x(t)] = 0$$

$$R_x(t, u) = \alpha^2 \min(t, u) + \frac{\beta^2 V_o}{2}(t^{2\rho} + u^{2\rho} - |t - u|^{2\rho})$$

Then, since

$$\tau_n = \frac{nN}{f_o} + x(\frac{nN}{f_o})$$

the second order statistics for (τ_n) are just

$$E(\tau_n) = (\frac{N}{f_o})n$$

$$E(\tau_n \tau_m) = (\frac{\alpha^2 N}{f_o})\min(m, n) + (\frac{\beta^2 V_o}{2})(\frac{N}{f_o})^{2\rho}(m^{2\rho} + n^{2\rho} - |m - n|^{2\rho}) + (\frac{N}{f_o})^2 mn$$

$$\mathrm{Var}(\tau_n) = (\frac{\alpha^2 N}{f_o})n + (\beta^2 V_o)(\frac{N}{f_o})^{2\rho} n^{2\rho}$$

Finally, since N/f_o will be on the order of one second, and since experimental results indicate that the "1/f" component of the variance dominates only after one second, we can assume that

$$\beta^2 V_o (\frac{N}{f_o})^{2\rho} \leq \frac{\alpha^2 N}{f_o}$$

Thus the statistics may be written as:

$$E(\tau_n) = n\bar{\tau}$$

$$Var(\tau_n) = \sigma_1^2 n + \sigma_2^2 n^{2\rho}$$

$$E(\tau_n \tau_m) = \sigma_1^2 \min(m, n) + \frac{\sigma_2^2}{2} (m^{2\rho} + n^{2\rho} - |m - n|^{2\rho}) + \bar{\tau}^2 mn$$

where

$$\sigma_1^2 \geq \sigma_2^2$$

and

$$\frac{1}{2} < \rho < 1$$

APPENDIX H

SIMULTANEOUS ESTIMATION OF RANDOM AND NONRANDOM VECTORS

In this Appendix we consider the following problem. Let x be a random vector of dimension p with mean vector and covariance matrix given by

$$\mu_x = E(x)$$

$$P_x = E[(x-\mu_x)(x-\mu_x)']$$

Also, let y be a non-random vector of dimension q; that is, we have no a priori information, either statistical or set theoretic, about y. Given an observation vector z, of dimension r, of the form

$$z = F\,x + Gy + n \tag{H.1}$$

we wish to estimate the values of x and y. In the above, F and G are matrices of the appropriate dimensions, and n is a noise random vector with mean and covariance

$$\mu_n = E(n)$$

$$P_n = E[(n-\mu_n)(n-\mu_n)']$$

Initially, the random vectors x and n will be allowed to have arbitrary cross-covariance

$$P_{nx} = E[(n - \mu_n)(x - \mu_x)'] = P'_{xn}$$

We shall eventually specialize to the case where $P_{nx} = 0$.

In order to insure a nonsingular estimation problem we shall assume that

$$P_n + F\,P_x F' + P_{nx}\,F' + F\,P_{xn} > 0 \tag{H.2}$$

i.e., that the above matrix is positive definite. This will be true, when we specialize to the case where $P_{nx} = 0$, if $P_n > 0$. Also, in order to insure a unique optimal estimate of y, we shall assume that

$$\text{col rank }(G) = q \tag{H.3}$$

where col rank (G) is the number of linearly independent columns of G, and q is the dimension of y.

We begin with the derivation for the estimate of y. Defining the random vector ε as

$$\varepsilon = n + F\,x$$

it follows that the mean and covariance of ε are

$$\mu_\varepsilon = \mu_n + F\mu_x$$

$$P_\varepsilon = P_n + FP_xF' + P_{nx}F' + FP_{xn}$$

Then, (H.1) can be rewritten as

$$z = Gy + \varepsilon$$

Since y is a non-random vector, a reasonable estimate for y is the least-squares estimate, i.e., the estimate $\hat{y}$ which minimizes

$$(z - G\hat{y} - \mu_\varepsilon)'\ P_\varepsilon^{-1}(z - G\hat{y} - \mu_\varepsilon) \tag{H.4}$$

If x and n were jointly Gaussian random vectors, then this estimate would also be the maximum likelihood estimate of y, i.e., the estimate consistent with the observation z and the most probably value of the random vector ε.

The unique vector $\hat{y}$ that minimizes (H.4) is just .

$$\hat{y} = K\ (z - \mu_\varepsilon) \tag{H.5}$$

where

$$K = (G'P_\varepsilon^{-1}G)^{-1}G'\ P_\varepsilon^{-1} \tag{H.6}$$

The matrix K is well defined because of assumptions (H.2) and (H.3). If (H.3) were relaxed, the set of $\hat{y}$ which minimizes (H.4) would be a linear manifold of dimension ≥ 1, and we could define $\hat{y}$ to be the unique vector of minimum norm in this manifold.

Denoting the error in the estimate $\hat{y}$ as

$$\xi_y = \hat{y} - y$$

it follows that

$$\begin{aligned}\xi_y &= K\,(z - \mu_\varepsilon) - y \\ &= K\,(Gy + \varepsilon - \mu_\varepsilon) - y \\ &= K\,(\varepsilon - \mu_\varepsilon) + (KG - I)\,y\end{aligned}$$

However, from (H.6) we have

$$KG - I = 0$$

so that

$$\xi_y = K\,(\varepsilon - \mu_\varepsilon) \qquad \text{(H.7)}$$

Therefore, it is clear that the estimate $\hat{y}$ is unbiased and has an error covariance matrix

$$P_{\xi_y} = K\,P_\varepsilon\,K'$$

Using (H.6), the above reduces to

$$P_{\xi_y} = (G'P_\varepsilon^{-1}G)^{-1} \tag{H.8}$$

We now begin the derivation for the estimate of x by rewriting (H.1) as

$$z = G(y - \hat{y} + \hat{y}) + Fx + n$$

or,

$$z - G\hat{y} = -\,G\xi_y + Fx + n$$

Then, using (H.5), (H.7), and the definition of ε,

$$z - GK\,(z - \mu_\varepsilon) = Fx + n - GK(Fx + n - \mu_\varepsilon)$$

or

$$(I - GK)z = (I - GK)\,(Fx + n) \tag{H.9}$$

It is worth noting at this point that an equation equivalent to (H.9) could have been derived directly from (H.1) as follows. Since we are interested in estimating x, and since y is completely unknown, a reasonable approach is to remove the effects of y on (H.1) by premultiplying both sides of (H.1) by a matrix M for which

$$\text{Ker}\,(M) = \text{Im}(G) \tag{H.10}$$

where Ker (·) denotes the kernel, or null space, and Im (·) denotes the image space, or range. For such an M, (H.1) then yields

$$Mz = MFx + Mn$$

and we have a completely statistical estimation problem. It is easy to see that the matrix (I-GK) satisfies Ker (I-GK) = Im (G).

We may also note that if rank (G) = r, the dimension of z, then the only matrix M satisfying (H.10) is the zero matrix. In this case, the unknown vector y completely masks x, and the only possible estimate for x is the best a priori estimate, μ_x.

Resuming the derivation for the estimate of x, define

$$\begin{aligned} w = (I - GK)z &= (I - GK)\,(Fx + n) \\ &= (I - GK)\varepsilon \end{aligned} \tag{H.11}$$

Then, by standard procedures, the linear minimum-mean-square-error estimate of x would be just

$$\hat{x} = P_{xw} P_w^{-1} (w - \mu_w) + \mu_x$$

if P_w were nonsingular. However, from (H.11)

$$P_w = (I - GK)\, P_\varepsilon\, (I-GK)'$$

and since Ker (I - GK) = Im (G), it is clear that (I-GK), and thus P_w also, is singular.

The singularity of this estimation problem is explained by the fact that, since (I - GK) is singular, the vector w contains redundant information. This redundancy may be removed as follows. Let M_0 be any matrix of full row rank satisfying (H.10); M_0 will be (r - q) by r. Since Ker (I - GK) = Ker (M_0), there exists a matrix T(r by (r - q)) such that

$$I - GK = TM_0 \qquad (H.12)$$

Now define a vector $\hat{w}$ as

$$\hat{w} = M_0\, z$$

The vectors w and $\hat{w}$ are related by

$$w = T\,\hat{w}$$

$$\hat{w} = (T'T)^{-1}T'\,w$$

In light of (H.10), we have

$$\hat{w} = M_o z = M_o(Fx + n) = M_o\varepsilon \qquad \text{(H.13)}$$

so that the estimate for x is just

$$\hat{x} = P_{x\hat{w}}\,P_{\hat{w}}^{-1}\,(\hat{w} - \mu_{\hat{w}}) + \mu_x$$

From (H.13), the above becomes

$$\hat{x} = P_{x\varepsilon}\,M'_o(M_o P_\varepsilon M'_o)^{-1}M_o(z - \mu_\varepsilon) + \mu_x$$

Since M_o has full row rank, and P_ε is assumed positive definite, the above matrix inverse does indeed exist. However, $\hat{x}$ is written in terms of M_o, a matrix that we would just as soon not have to actually find. To remove this dependence, we need the following proposition.

<u>Proposition</u>: $M_0' \, (M_0 P_\varepsilon M_0')^{-1} M_0 = P_\varepsilon^{-1}(I - GK)$

<u>Proof</u>: Since $M_0 G = 0$,

$$M_0' = P_\varepsilon^{-1} \, (I - GK) \, P_\varepsilon M_0'$$

$$= P_\varepsilon^{-1} \, T M_0 P_\varepsilon M_0'$$

Using (H.12) therefore,

$$M_0'(M_0 P_\varepsilon M_0')^{-1} \, M_0 = P_\varepsilon^{-1} T M_0 = P_\varepsilon^{-1} \, (I - GK)$$

Using the Proposition, we now get as an estimate for x

$$\hat{x} = P_{x\varepsilon} \, P_\varepsilon^{-1}(I - GK) \, (z - \mu_\varepsilon) + \mu_x$$

and, using the definition of ε, this becomes

$$\hat{x} = H\,(z - \mu_\varepsilon) + \mu_x \tag{H.14}$$

where

$$H = (P_x F' + P_{xn})\, P_\varepsilon^{-1}(I - GK) \tag{H.15}$$

The error in this estimate is

$$\xi_x = \hat{x} - x$$

and the covariance matrix of ξ_x is, by standard techniques,

$$\begin{aligned} P_{\xi_x} &= P_x - P_{x\hat{w}} P_{\hat{w}}^{-1} P_{\hat{w}x} \\ &= P_x - P_{x\varepsilon} M_o' \,(M_o P_\varepsilon M_o')^{-1} M_o P_{\varepsilon x} \end{aligned}$$

Then, using the Proposition and (H.6),

$$P_{\xi_x} = P_x - (P_x F' + P_{xn})\, [P_\varepsilon^{-1} - P_\varepsilon^{-1} G(G' P_\varepsilon^{-1} G)^{-1} G' P_\varepsilon^{-1}](F P_x + P_{nx}) \tag{H.16}$$

The expressions for $\hat{x}$ and $\hat{y}$ given by (H.14) and (H.5) and the error covariance matrices given by (H.16) and (H.8) may be simplified somewhat when we specialize to the case where x and n are uncorrelated. Then,

$$P_\varepsilon = P_n + FP_xF'$$

We shall make use of the following lemma several times.

Lemma: Let A, B, and C be arbitrary matrices such that A^{-1} exists. Then $(A+BC)^{-1}$ exists if and only if $(I+CA^{-1}B)^{-1}$ exists and, moreover,

$$(A+BC)^{-1} = A^{-1}-A^{-1}B(I+CA^{-1}B)^{-1}CA^{-1}$$

Theorem: Under the conditions

$P_x>0$

$P_n>0$

$P_{nx}=0$

col rank $(G) = q$

the problem of estimating x and y is nonsingular, and the optimal extimates are given as

$$x = Hz + (I-HF)\mu_x - H\mu_n \tag{H.17}$$

$$y = Kz - KF\mu_x - K\mu_n \tag{H.18}$$

where H and K are

$$K = [G'(P_n+FP_xF')^{-1}G]^{-1}G'(P_n+FP_xF')^{-1} \tag{H.19}$$

$$H = P_xF'(P_n+FP_xF')^{-1}(I-GK) \tag{H.20}$$

Moreover, the error covariance matrices for these estimates are

$$P_{\xi_x} = [P_x^{-1}+F'P_n^{-1}F-F'P_n^{-1}G(G'P_n^{-1}G)^{-1}G'P_n^{-1}F]^{-1} \tag{H.21}$$

$$P_{\xi_y} = (G'P_n^{-1}G)^{-1} + (G'P_n^{-1}G)^{-1}G'P_n^{-1}FP_{\xi_x}F'P_n^{-1}G(G'P_n^{-1}G)^{-1} \tag{H.22}$$

Proof: First, since $P_n>0$,

$$P_\varepsilon = P_n + FP_xF' > 0$$

and we can apply the Lemma to compute P_ε^{-1}:

$$P_\varepsilon^{-1} = P_n^{-1} - P_n^{-1}F(P_x^{-1} + F'P_n^{-1}F)^{-1}F'P_n^{-1} \tag{H.23}$$

A second equation that will be useful is also obtained from the Lemma:

$$\begin{aligned}(P_x^{-1} + F'P_n^{-1}F)^{-1} &= P_x - P_xF'(P_n + FP_xF')^{-1}FP_x \\ &= P_x - P_xF'P_\varepsilon^{-1}FP_x \end{aligned} \tag{H.24}$$

A third equation is derived as follows:

$$P_n = P_\varepsilon - FP_xF'$$

so that

$$P_\varepsilon^{-1} = (I - P_\varepsilon^{-1}FP_xF')P_n^{-1}$$

Therefore,

$$\begin{aligned}P_xF'P_\varepsilon^{-1}G &= P_xF'(I - P_\varepsilon^{-1}FP_xF')P_n^{-1}G \\ &= (P_x - P_xF'P_\varepsilon^{-1}FP_x)F'P_n^{-1}G \\ &= (P_x^{-1} + F'P_n^{-1}F)^{-1}F'P_n^{-1}G \end{aligned} \tag{H.25}$$

where the last line follows from (H.24).

Now apply the Lemma to the right-hand side of (H.21) to get

$$[P_x^{-1}+F'P_n^{-1}F-F'P_n^{-1}G(G'P_n^{-1}G)^{-1}G'P_n^{-1}F]^{-1} =$$

$$(P_x^{-1}+F'P_n^{-1}F)^{-1}+(P_n^{-1}+F'P_n^{-1}F)^{-1}F'P_n^{-1}G[G'P_n^{-1}G-G'P_n^{-1}F(P_x^{-1}+F'P_n^{-1}F)^{-1}F'P_n^{-1}G]^{-1}$$

$$\cdot\ G'P_n^{-1}F(P_x^{-1}+F'P_n^{-1}F)$$

$$= (P_x-P_xF'P_\varepsilon^{-1}FP_x) + (P_xF'P_\varepsilon^{-1}G)(G'P_\varepsilon^{-1}G)^{-1}(G'P_\varepsilon^{-1}FP_x)$$

where each of the bracketed terms in the last line follows from one of (H.23), (H.24), or (H.25). However, this last line is precisely the expression for P_{ξ_x} given by (H.16) for the case where $P_{nx} = 0$. This proves (H.21).

To prove (H.22), we have from (H.8) and (H.23),

$$P_{\xi_y} = (G'P_\varepsilon^{-1}G)^{-1}$$

$$= [G'P_n^{-1}G - G'P_n^{-1}F(P_x^{-1} + F'P_n^{-1}F)^{-1}F'P_n^{-1}G]^{-1}$$

Applying the Lemma to this inverse,

$$P_{\xi_y} = (G'P_n^{-1}G)^{-1} +$$

$$(G'P_n^{-1}G)^{-1}G'P_n^{-1}F[P_x^{-1}+F'P_n^{-1}F-F'P_n^{-1}G(G'P_n^{-1}G)^{-1}G'P_n^{-1}F]^{-1}$$

$$\cdot F'P_n^{-1}G(G'P_n^{-1}G)^{-1}$$

Substitution of (H.21) into the above proves (H.22).

Equations (H.17) through (H.20) follow immediately from (H.5), (H.6), (H.14), and (H.15).

<u>Corollary</u>: *Under the conditions of the Theorem, and with*

$$P_n = \sigma^2 I$$

the error covariance matrices are just

$$P_{\xi_x} = \sigma^2[\sigma^2 P_x{}^{-1} + F'F - F'G(G'G)^{-1}G'F]^{-1}$$

$$P_{\xi_y} = \sigma^2(G'G)^{-1} + (G'G)^{-1}G'FP_{\xi_x}F'G(G'G)^{-1}$$

A second corollary to the Theorem can be obtained for the case where x is a scalar and $P_n = \sigma^2 I$. For this case, F is a column vector.

<u>Corollary</u>: *Under the conditions of the Theorem, and with*

$$P_n = \sigma^2 I$$

and

$$p = 1 \quad \text{(i.e., x is a scalar)}$$

the estimates $\hat{x}$ and $\hat{y}$ are given by

$$\hat{x} = \gamma F'(I - G(G'G)^{-1}G')\ (z - F\mu_x - \mu_n) + \mu_x \tag{H.26}$$

$$\hat{y} = (G'G)^{-1}G'(z-Fx-\mu_n) \tag{H.27}$$

where

$$\gamma = \frac{\sigma_x^2}{\sigma^2+\sigma_x^2[F'F-F'G(G'G)^{-1}G'F]}$$

Proof: The above expressions follow from (H.17) and (H.18). The steps are straightforward but detailed, and are thus left out.

We close the Appendix with the following observations. If F and G are constrained so that (F'F) and (G'G) are held fixed, but F'G is allowed to vary, then, for the case where

$$P_n = \sigma^2 I$$

$$P_{nx} = 0$$

we should choose

$$F'G = 0$$

to obtain the best estimator performances.

REFERENCES

1. I. G. Stiglitz, J. U. Beusch, A. E. Eckberg, and K. S. Schneider, "Concept Formulation Studies of the Surveillance Aspects of the Fourth Generation Air Traffic Control System," Lincoln Laboratory Project Report ATC-7, (21 September 1971).

2. S. O. Rice, "Mathematical Analysis of Random Noise," BSTJ, XXIII, July, 1944, No. 3, 282-332.

3. R. G. Gallager, "Lower Bounds On the Tails of Probability Distributions," M.I.T. Research Laboratory of Electronics, Quarterly Progress Report No. 77, 281.

4. G. R. Cooper and R. D. Yates, Design of Large Signal Sets With Good Aperiodic Correlation Properties, Purdue University School of Electrical Engineering Electrical Systems Research Laboratory, Lafayette, Indiana, (September 1966) 14.

5. IEEE Proceedings, Special Issue on Frequency Stability, (February 1966).

6. B. Mandelbrot and J. Van Ness, "Fractional Brownian Motions, Fractional Noises and Applications," SIAM Review, 10, 422-437, (October 1968).

7. R. S. Burington, Handbook of Mathematical Tables and Formulas, Handbook Publishers, Sandusky, Ohio (1948).