# Implementing the Matrix Exponential Function on Embedded Processors

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# Definition
## [Moler and Van Loan,2003]

**The solution to the differential equation**

$$\dot{x} = Ax(t)$$

$$x(0) = x_0$$

**is given by**

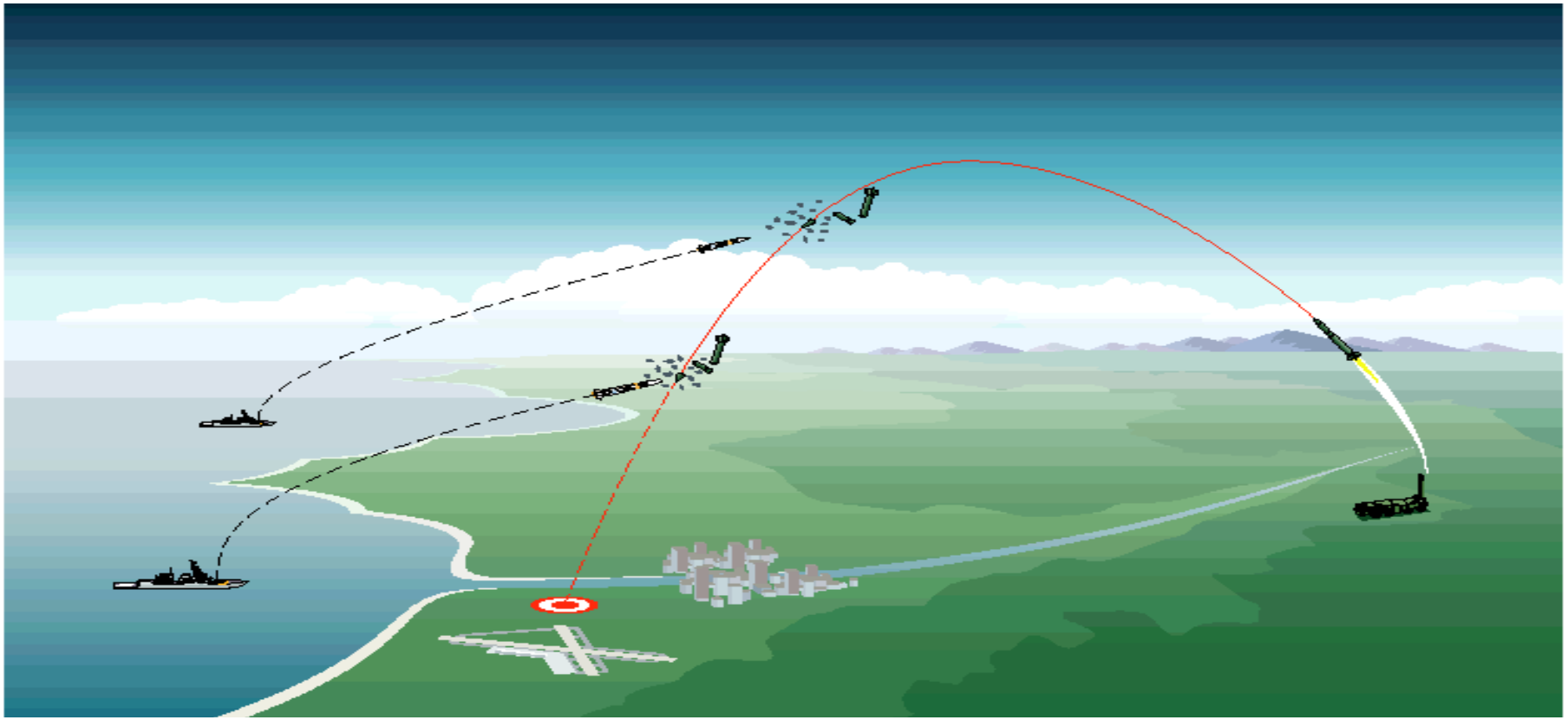
$$x(t) = e^{At} x_0$$

**Where** $e^{At}$ **is the matrix exponential function,**

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \ldots$$

**Notice that if** $A = [a_{ij}], e^{At} \neq [e^{a_{ij} t}]$ **in general.**

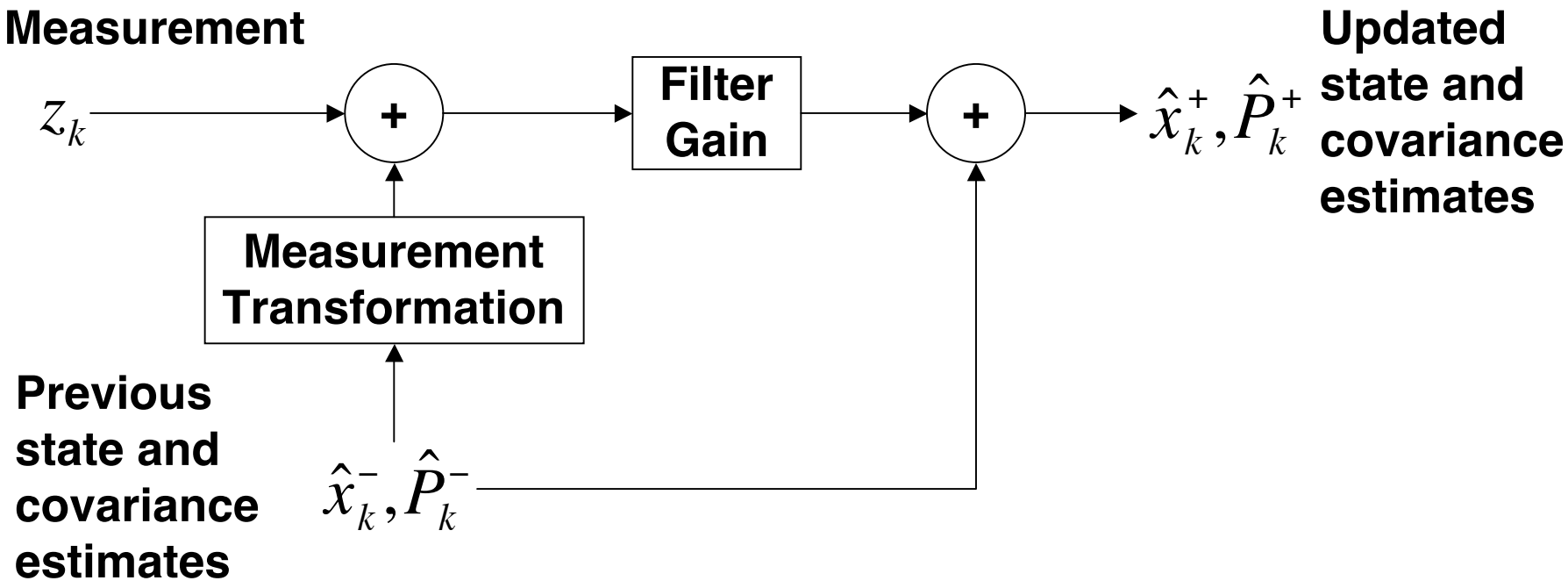
# Application: Ballistic Target Tracking



- Tracking of a ballistic target using noisy measurements
- Tracking accomplished using the *extended Kalman filter*
  - "extended" means that system dynamics are non-linear

# The Extended Kalman Filter

**Estimate next state based on previous state and new measurement**




$$\hat{P}_k^+ = \Phi \hat{P}_k^- \Phi + Q(t),$$

**where** $\Phi = e^{Jt}$ **for a matrix** $J$**,**

**and** $Q(t)$ **is the process noise covariance.**

# Calculation Overview

**Preferred method, Padé approximation, is only valid when $||A||$ is small**

**Use the fact that $e^{A} = (e^{A/m})^{m}$**

1. **Choose an integer $j$ and scale $A$ by $m=2^j$**
2. **Use a Padé approximation to calculate $E = e^{A/2^j}$**
3. **Perform $j$ matrix multiplies to calculate $E^{2^j}$**

**This technique is referred to as "scaling and squaring" [4,5].**

# Padé Iteration Algorithm

```
X = A;
c = 1;
E = I;
D = I;
for(k = 1; k <= q; k++) // q=number of iterations
{
   c = c * (q-k+1) / (k*(2*q-k+1));
   X = A*X;                  // Matrix multiply
   E = E + cX;               // Matrix scale and add
   if (k is even)            // Matrix add or subtract
     D = D + cX;
   else
     D = D - cX;
}
E = D\E;                     // Solve using LU factorization
```

# Implementation Overview

| Step | Operations | Percentage of op count |
|---|---|---|
| Scale the matrix $A$ | Elementwise multiply | <2% |
| Padé iteration | Matrix multiply, scale, add | 50-75% |
| LU and backsolve | | 3-6% |
| Repeated squaring | Matrix multiply | 13-50% |

Op counts assume 6 Padé iterations

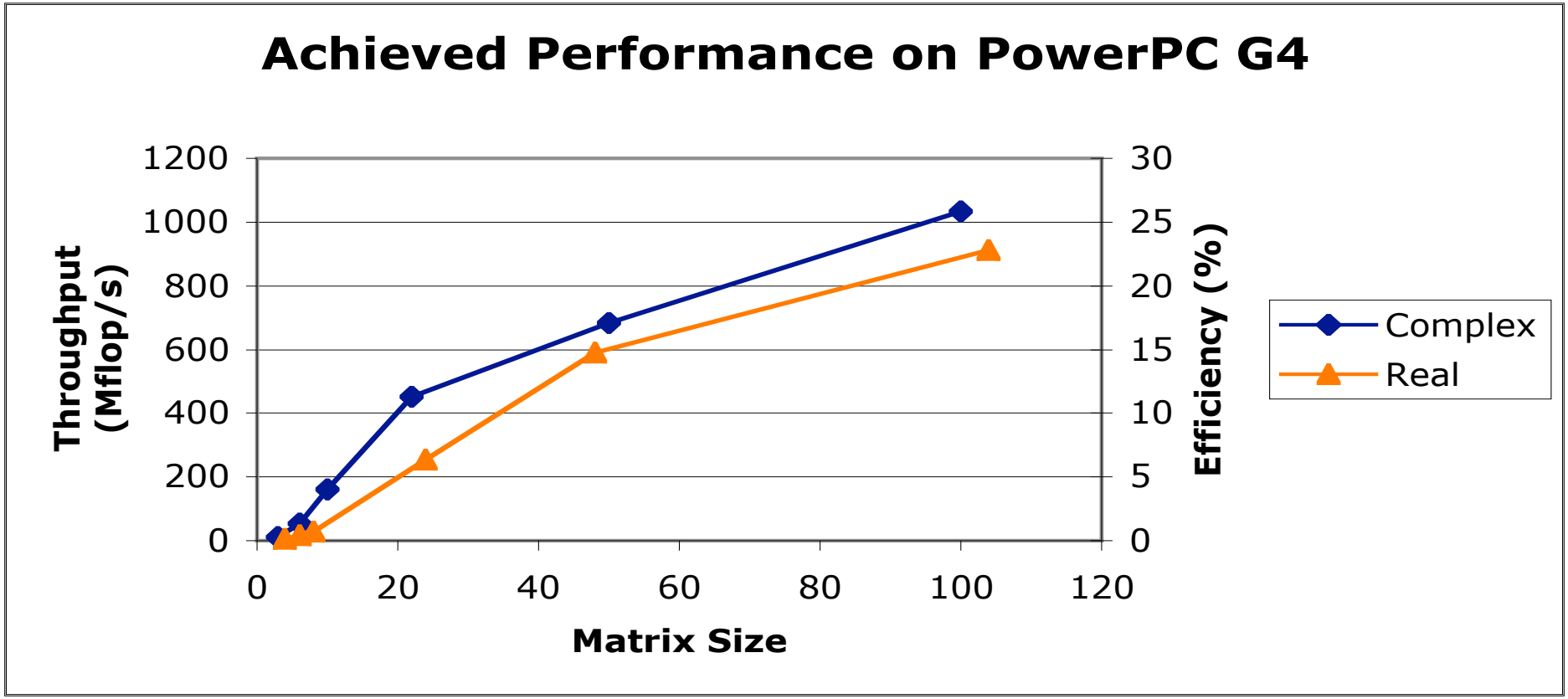
**Implementation Features**

- Single-precision real or complex float
- C++
- Uses an object for storage
- Calls VSIPL routines
- Uses Altivec-optimized matrix multiply
- Choose accuracy to match limits of single-precision calculations

```
void create(Matrix<T> &A,
            Matrix<T> &E);
void run(   Matrix<T> &A,
            Matrix<T> &E);
```

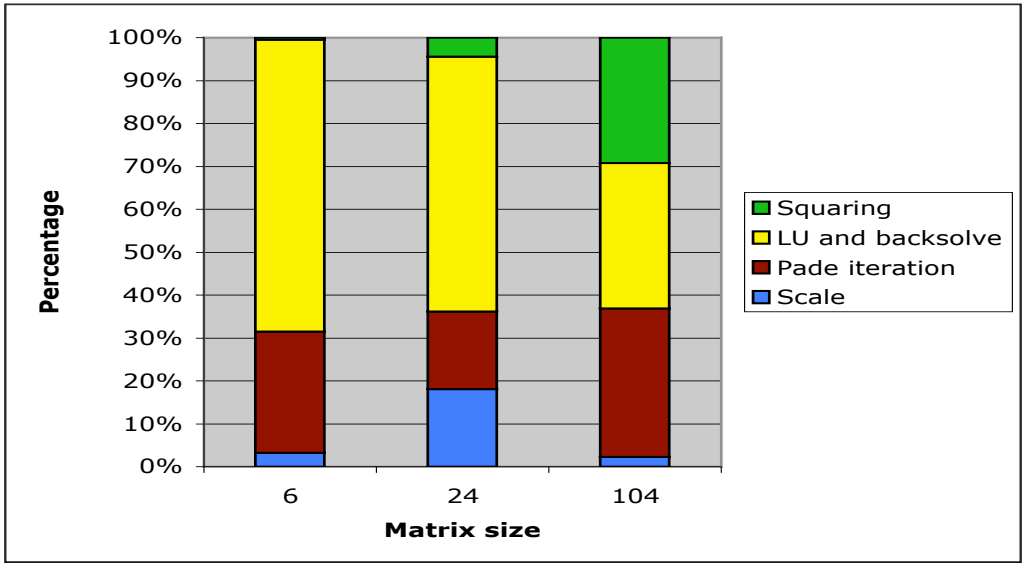
```
// Allocates memory & initializes
// LU factorization
// Performs computation
```

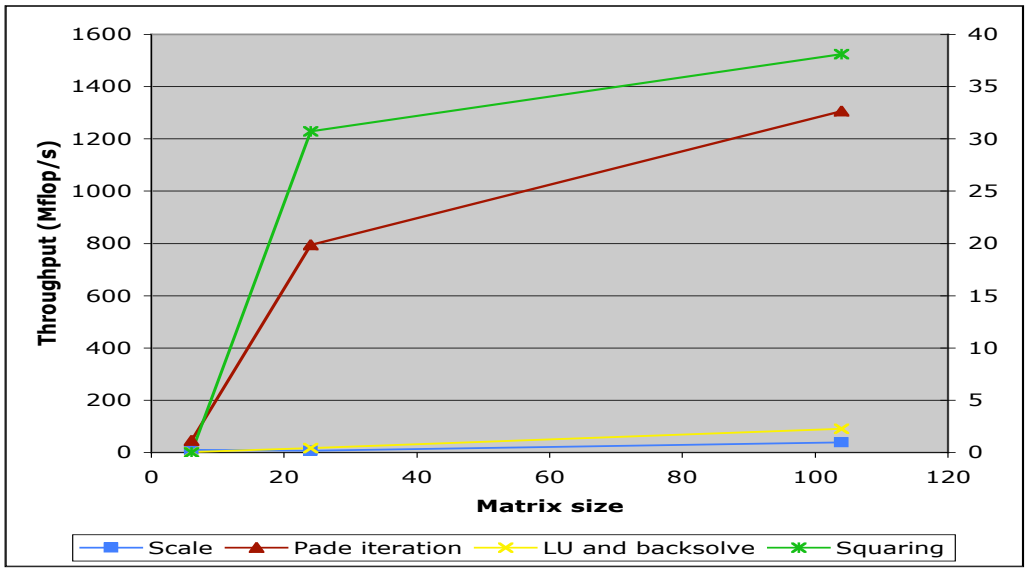
# Performance




- Platform: Mercury 500 MHz PowerPC G4
- Achieves respectable performance for large matrices
- For tracking, sizes of interest are small — 6x6 matrices
  - A tuned implementation could be produced for this size

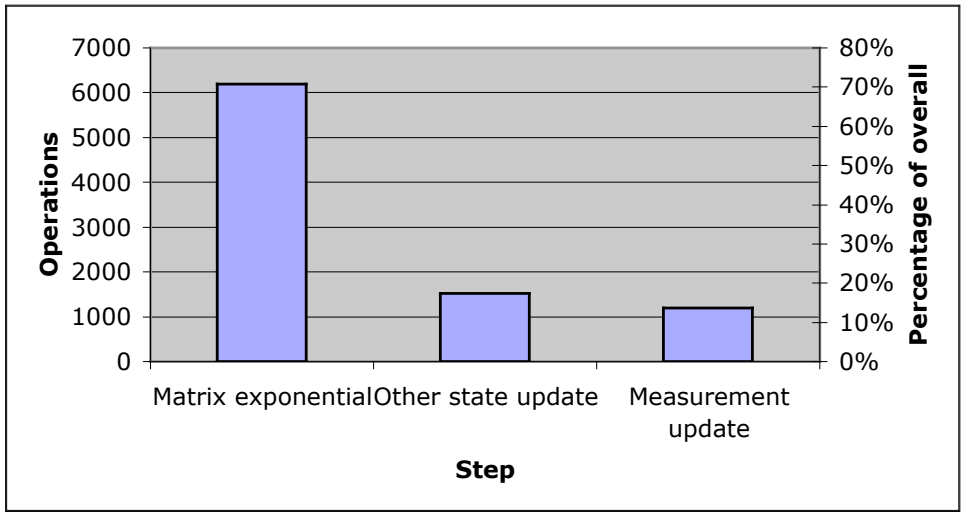
# Performance Breakdown

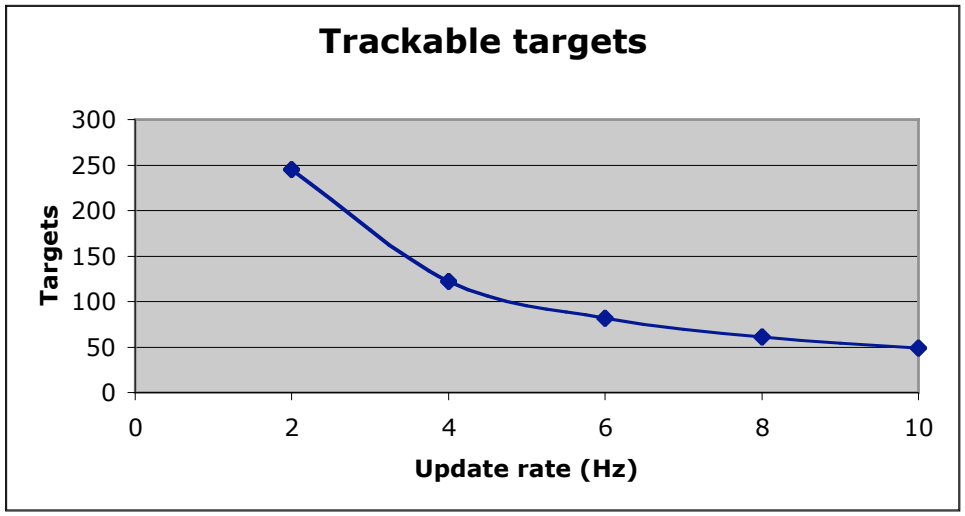



- Performance breakdown on PowerPC G4
- Steps based on matrix multiply are more efficient than other steps
- For large matrices, matrix multiply steps still consume most of the execution time
- LU/backsolve is a substantial percentage of time despite being a low percentage of the op count

# The Matrix Exponential in Tracking




- Matrix exponential is a substantial part of the EKF's operation count
- How many targets could a single processor track?
  - Assume 500 MHz PPC G4
  - Use execution time of 6x6 real matrix exponential
  - Assume remainder of EKF has efficiency comparable to LU factorization (~0.04%)
  - Vary track rate from 2-10 Hz
- A single processor can potentially track many targets

# Conclusions

- Matrix exponential function is important for tracking applications
- A large percentage of the operations are matrix multiply functions
- An efficient implementation of this function allows it to be used in an extended Kalman filter
- Many targets can be tracked using even a single processor
  - Using multiple processors obviously allows more targets to be tracked

# References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