Optimizing the Fast Fourier Transform over Memory Hierarchies for Embedded Digital Systems: A Fully In-Cache Algorithm

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New FFT algorithm for embedded systems

- **Maximize in-cache operations through use of repeated transpose-reshape operations**
- **Similar to partitioning for parallel implementation**
- **Do as many operations in cache as possible**
- **Re-materialize the array to achieve locality**
- **Continue processing in cache and repeat process**

Example

- **Assume cache size c = 4; input vector length n = 32; number of rows r = n/c = 8**

- **Generate vector of indices:**

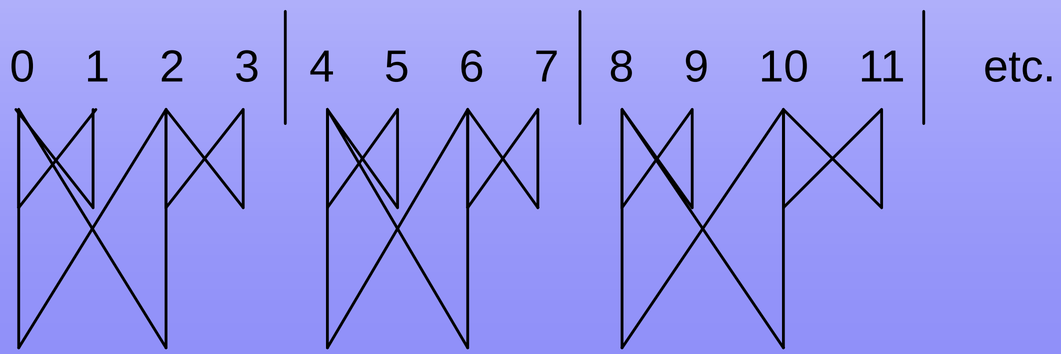
$$v = i(n) = < 0\ 1\ 2 \ldots 31 >$$

- **Use re-shape operator r to generate a matrix**

Starting Matrix

- **Each row is of length equal to the cache size**
- **Standard butterfly applied to each row as...**

$$A \equiv \langle rc \rangle \hat{\rho} v = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$



Next transpose

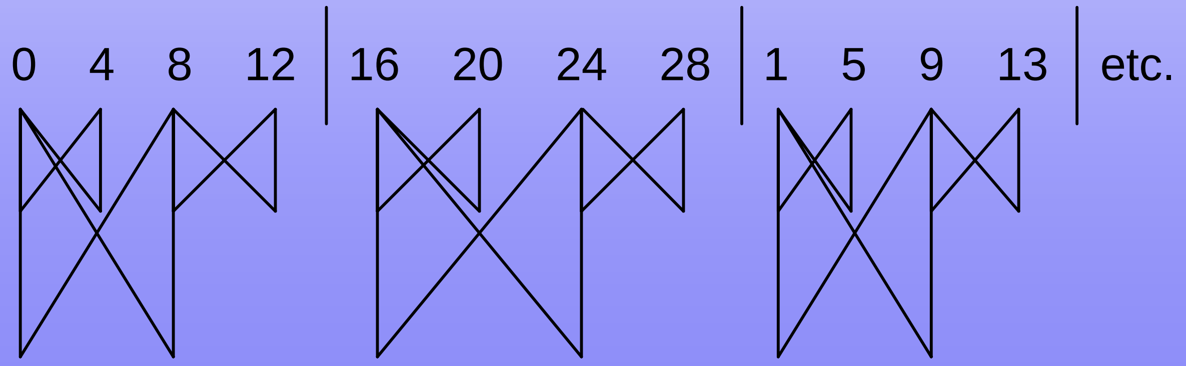
- **To continue further would induce cache misses so transpose and reshape.**
- **Transpose-reshape operation composed over indices (only result is materialized.**
- **The transpose is:**

$$A^T = \begin{bmatrix} 0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 \\ 1 & 5 & 9 & 13 & 17 & 21 & 25 & 29 \\ 2 & 6 & 10 & 14 & 18 & 22 & 26 & 30 \\ 3 & 7 & 11 & 15 & 19 & 23 & 27 & 31 \end{bmatrix}$$

Resulting Transpose-Reshape

- **Materialize the transpose-reshaped array B**
- **Carry out butterfly operation on each row**
- **Weights are re-ordered**
- **Access patterns are standard...**

$$B \equiv \langle rc \rangle \hat{\rho}(A^T) = \begin{bmatrix} 0 & 4 & 8 & 12 \\ 16 & 20 & 24 & 28 \\ 1 & 5 & 9 & 13 \\ 17 & 21 & 25 & 29 \\ 2 & 6 & 10 & 14 \\ 18 & 22 & 26 & 30 \\ 3 & 7 & 11 & 15 \\ 19 & 23 & 27 & 31 \end{bmatrix}$$



Transpose-Reshape again

- **As before: to proceed further would induce cache misses so:**
- **Do the transpose-reshape again (composing indices)**
- **The transpose is:**

$$B^T = \begin{bmatrix} 0 & 16 & 1 & 17 & 2 & 18 & 3 & 19 \\ 4 & 20 & 5 & 21 & 6 & 22 & 7 & 23 \\ 8 & 24 & 9 & 25 & 10 & 26 & 11 & 27 \\ 12 & 28 & 13 & 29 & 14 & 30 & 15 & 31 \end{bmatrix}$$

Last step (in this example)

- **Materialize the composed transpose-reshaped array C**
- **Carry out the last step of the FFT**
- **This last step corresponds to cycles of length 2 involving elements 0 and 16, 1 and 17, etc.**

$$C \equiv \langle rc \rangle \hat{\rho}(B^T) = \begin{bmatrix} 0 & 16 & 1 & 17 \\ 2 & 18 & 3 & 19 \\ 4 & 20 & 5 & 21 \\ 6 & 22 & 7 & 23 \\ 8 & 24 & 9 & 25 \\ 10 & 26 & 11 & 27 \\ 12 & 28 & 13 & 29 \\ 14 & 30 & 15 & 31 \end{bmatrix}$$

Summary

- **All operations have been carried out in cache at the price of re-arranging the data**
- **Data blocks can be of any size (powers of the radix): need not equal the cache size**
- **Optimum performance: tradeoff between reduction of cache misses and cost of transpose-reshape operations**
- **Number of transpose-reshape operations determined by the data block size (cache size)**