High-Performance Code Generation for FIR Filters and the Discrete Wavelet Transform Using SPIRAL

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SPIRAL project
http://www.spiral.net

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Motivation

- **FIR filters - image enhancement, equalization, speech synthesis, biomedical signal processing, …**
- **Wavelets - image compression, detection, denoising, signal recovery**
- **Numerically intensive - significant impact on application performance**

High-Performance Implementations (How-To)

- **Choose Fast Algorithm**
  - **Arithmetic cost – only a rough estimate of performance**
  - **Many algorithms with similar cost – which one?**
- **Design Efficient Code**
  - **Architecture conscious – machine dependent**
  - **Obsolete when platform is changed/upgraded**
- **Best implementation: Algorithm + Machine + Compiler**
  - **Requires experts in algorithms and computer architecture**
  - **Frequent re-implementation**

Better way: automatic performance tuning

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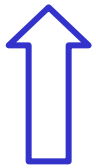
Existing FIR filtering and DWT software

Matlab©, Mathematica, S+Wavelets, WaveLab, Wave++, IPP

Application oriented – not machine specific, or

Hand-tuned code for specific platform

Automatic platform-adapted solutions not available !



We want to close this gap

Automatic performance tuning packages

ATLAS, PHiPAC, SPARSITY, FFTW, SPIRAL

Include several basic linear algebra and DSP functions

No filtering and wavelet kernels !

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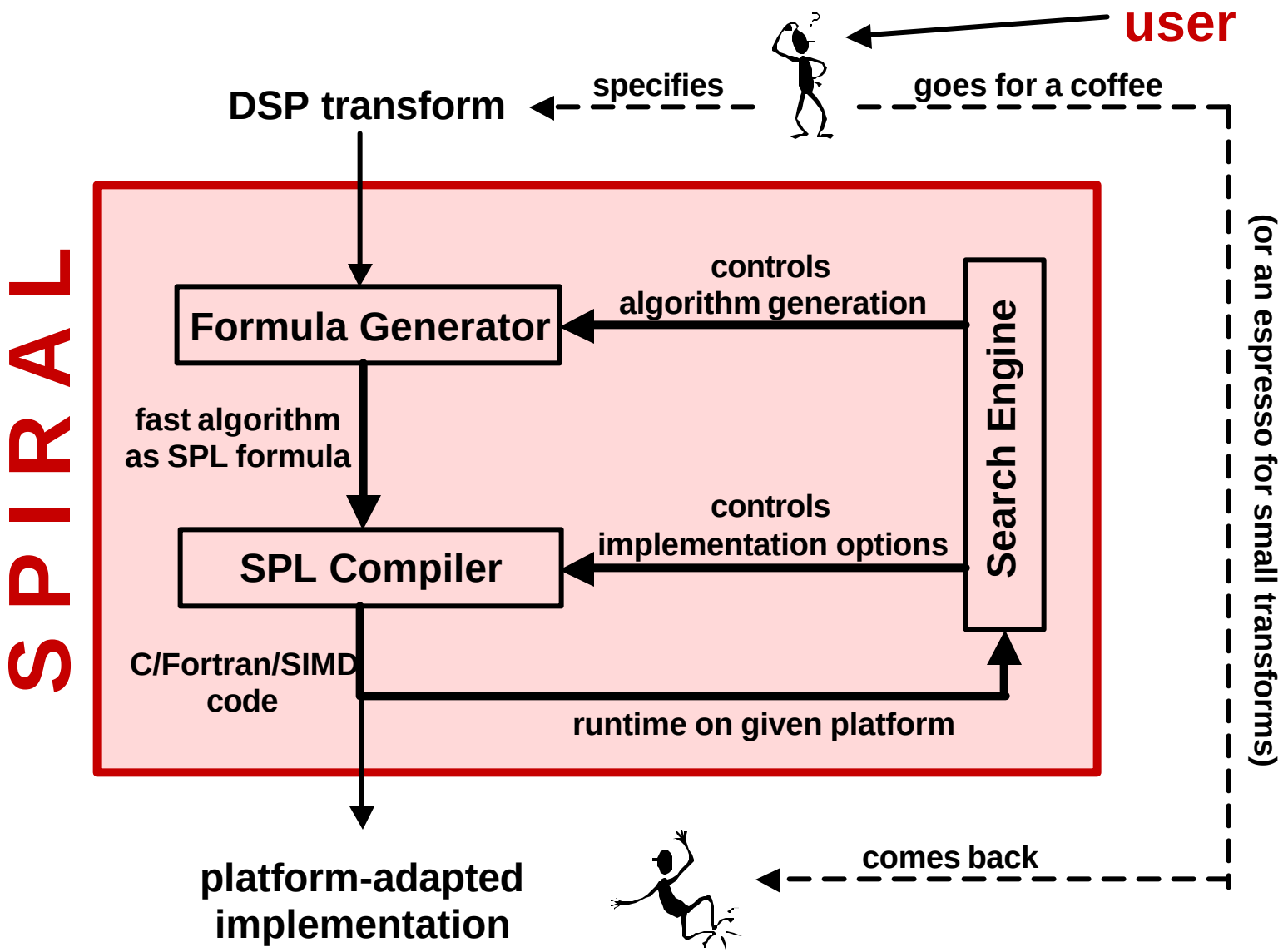
SPIRAL system

Generator of optimized DSP transform implementations



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SPIRAL's Four Key Concepts

Transform

$$\mathrm{DFT}_n = \left[e^{2\pi i kl/n} \right]_{k,l=0,\ldots,n-1}$$

parameterized matrix

Rule

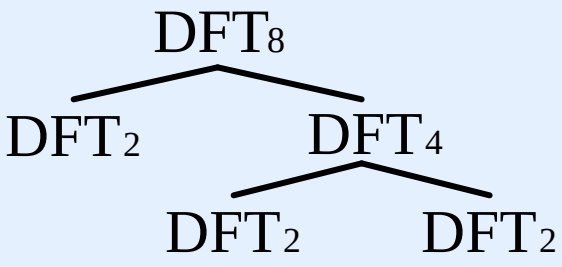
$$\mathrm{DFT}_{nm} = (\mathrm{DFT}_n \otimes I_m)\, T^{nm}_n\, (I_n \otimes \mathrm{DFT}_m)\, L^{nm}_m$$

Diagonal matrix (twiddles)

Kronecker product — Identity — Permutation

- a breakdown strategy - product of sparse matrices
- captures important structural information

Rule Tree



- recursive application of rules
- uniquely defines an algorithm
- efficient representation
- easy manipulation

Formula

$$\mathrm{DFT}_8 = (F_2 \otimes I_4)\, T^8_4\, (I_2 \otimes (I_2 \otimes F_2))\, L^8_2$$

- few constructs and primitives
- uniquely defines an algorithm
- can be translated into code

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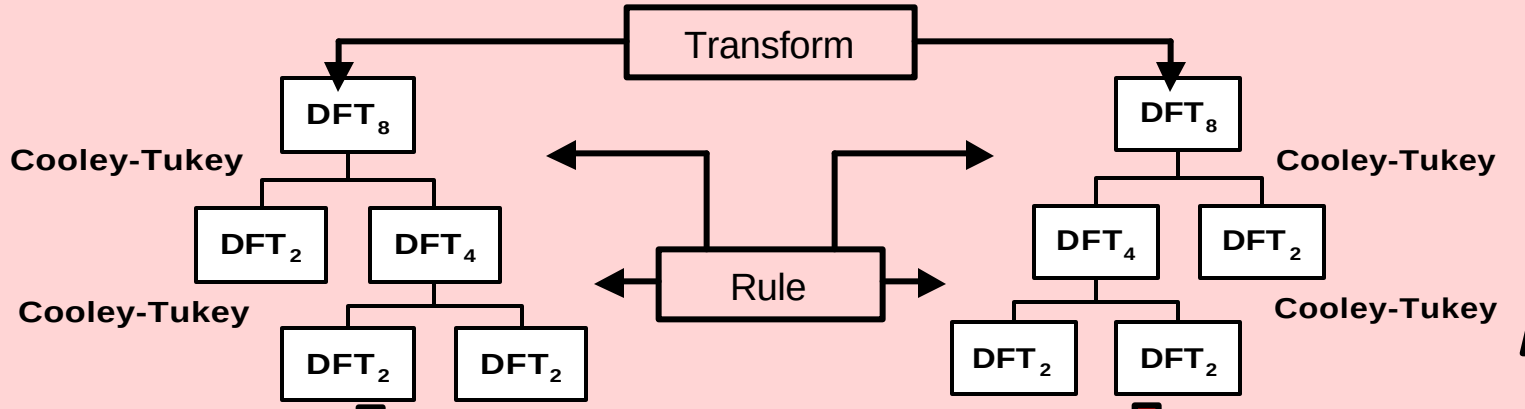
Rule Trees = Formulas = Algorithms

Rule Trees



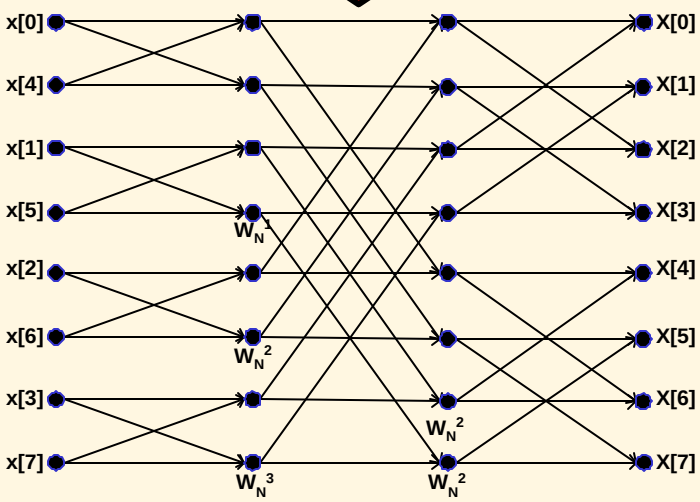
Formulas

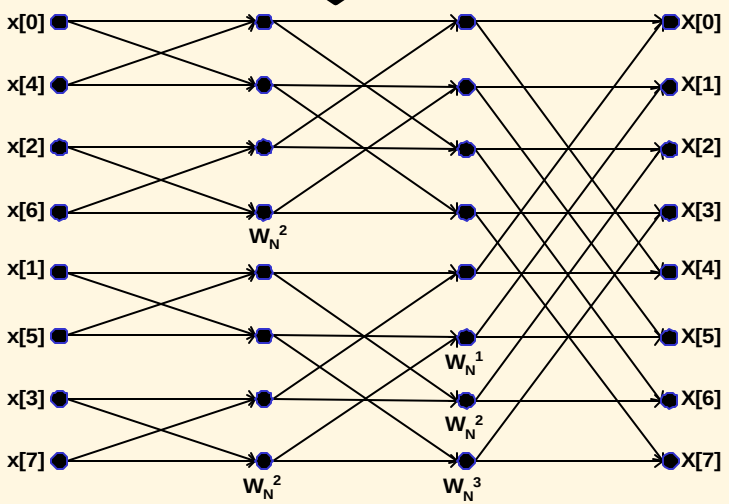
$$DFT_8 \;?\; (DFT_2 \;?\; I_4)\;?T_2^8\;?(I_2 \;?\; DFT_4)\;?L_4^8$$

$$DFT_8 \;?\; (DFT_4 \;?\; I_2)\;?T_4^8\;?(I_4 \;?\; DFT_2)\;?L_2^8$$

$$(DFT_2 \;?\; I_2)\;?T_2^4\;?(I_2 \;?\; DFT_2)\;?L_2^4$$

Algorithms





Increasing level of abstraction

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Examples: Rules and Rule Trees

Trigonometric Rule

$$\mathrm{DFT}_n \rightarrow \mathrm{CosDFT}_n + i \cdot \mathrm{SinDFT}_n$$

DCT-Recursive Rule

$$\mathrm{CosDFT}_n \rightarrow S_n \cdot \left(\mathrm{CosDFT}_{n/2} + \mathrm{DCT}^{(II)}_{n/4} \cdot S_n \cdot L^n_2\right)$$

$$\mathrm{SinDFT}_n \rightarrow S_n \cdot \left(\mathrm{SinDFT}_{n/2} + \mathrm{DCT}^{(II)}_{n/4} \cdot S_n \cdot L^n_2\right)$$

Type-Split Rule

$$\mathrm{DCT}^{(II)}_n \rightarrow P_n \cdot \left(\mathrm{DCT}^{(II)}_{n/2} + \mathrm{DCT}^{(IV)}_{n/2} \cdot P_n\right) \cdot \left(\mathrm{I}_{n/2} + F_2 \cdot P_n\right)$$

Type-Conversion Rule

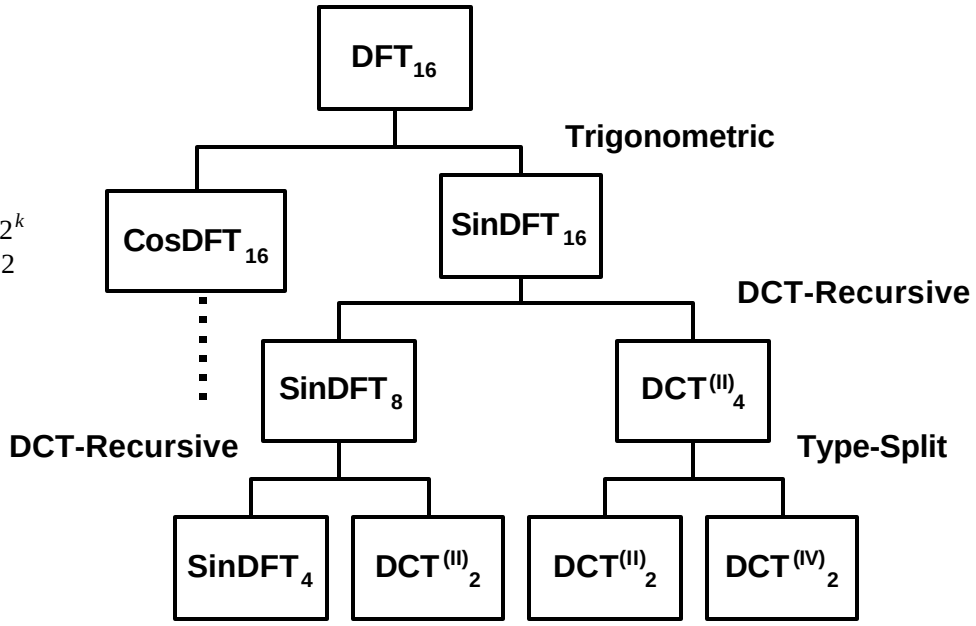
$$\mathrm{DCT}^{(IV)}_n \rightarrow S_n \cdot \mathrm{DCT}^{(II)}_n \cdot D_n$$

Recursive RHT Rule

$$\mathrm{RHT}_{2^k} \rightarrow \left(\mathrm{RHT}_{2^{k-1}} + \mathrm{I}_{2^{k-1}}\right) \cdot F_2 + \mathrm{I}_{2^{k-1}} \cdot L^{2^k}_2$$

Number of rule trees grows exponentially with

- ~~Size of the transform~~
- ~~Number of applicable rules~~



SPIRAL also includes search over implementation options, such as the degree of loop unrolling

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FIR Filters and the DWT in SPIRAL: Challenges

- Existing filtering and wavelet algorithm representations do not capture all important structural information
- SPIRAL uses a concise math framework to represent many algorithms for several transforms (DFT, DCTs, DHT, RDFT, …)
- Filtering and wavelet algorithms have considerably different structure from the current SPIRAL transforms
  - Current framework not enough to capture algorithms' structure
  - Existing algorithm representations need to be "spiralized" – i.e., captured concisely using the rule formalism
- Need changes on several levels:
  - New transforms, constructs, rules, translation templates
  - Modified search
  - Need to rephrase several concepts of the framework
- Existing framework has to be redesigned to accommodate new constructions and decomposition strategies

New framework has to be integrated in SPIRAL

FIR Filters: Rules and Algorithms

FIR Filter Transform

$$y_n = h_n * x_n = \sum_{i=0}^{k-1} h_i x_{n-i}$$

standard "sum" notation

$$F_n(\mathbf{h}) = \begin{bmatrix} h_0 & & & \\ h_1 & h_0 & & \\ \vdots & h_1 & \ddots & \\ h_{k-1} & \ddots & \ddots & h_0 \\ & h_{k-1} & & h_1 \\ & & \ddots & \vdots \\ & & & h_{k-1} \end{bmatrix}$$

SPIRAL transform notation

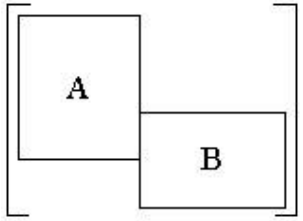
Baseline Rule

$$F_n(\mathbf{h}) = I_n \otimes_{k-1} \mathbf{h}^T$$

SPIRAL rule
captures the structure

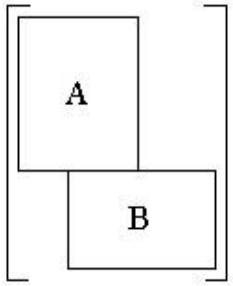
New constructs

$A \oplus^k B$

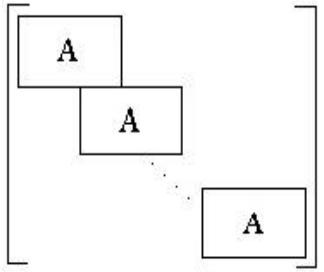


column and row overlapped direct sum

$A \oplus_k B$



$I_s \otimes^k A$



overlapped tensor product

Time Domain Methods

Overlap-Add Rule

$$F_n(\mathbf{h}) = I_{n/b} \otimes_{k-1} F_b(\mathbf{h})$$

concise rule notation

- Localized access of the input data
- Input is blocked into length b segments

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Overlap-Save Rule

$$F_n \left[\mathbf{h} \right] \; T \left[\mathbf{h}_L \right] \; \sum_{k=1} \; I_{m/b} \otimes \; _{k=1} F_b^T \left[\mathbf{h} \right] \; \sum_{k=1} \; T \left[\mathbf{h}_R \right]$$

Toeplitz matrix

- Localized storage of the output
- Output segments are computed independently

Blocking Rule

$$F_n \left[\mathbf{h} \right] \; I_{n/b} \otimes \; _{v} \sum_{i=1}^{k/b} \; ^{b} T \left[\mathbf{h}_i \right]$$

- Multilevel blocking
- Control over input & output locality

Frequency Domain Methods

Expanded Circulant Rule

$$F_n \left[\mathbf{h} \right] \; C \left[E_{k,n-1} \mathbf{h} \right] E_{n,k-1} \quad E_{n,k} \; \begin{bmatrix} I_n \\ 0_{k \times n} \end{bmatrix}$$

Cyclic-RDFT Rule

$$C \left[\mathbf{a} \right] \; \text{RDFT}_n^{-1} \; X \left[\text{RDFT}_n \mathbf{a} \right] \text{RDFT}_n$$

Circulant matrix **sparse matrix** **real DFT**

Radix-2 Cooley-Tukey RDFT Rule

$$\text{RDFT}_n \; \left[\text{RDFT}_{n/2} \; \otimes \; I_2 \right] X(w^{kl}) \left[I_{n/2} \; \otimes \; \text{RDFT}_2 \right] L_2^n$$

Time domain - better data access structure
Frequency domain - potentially reduce arithmetic cost

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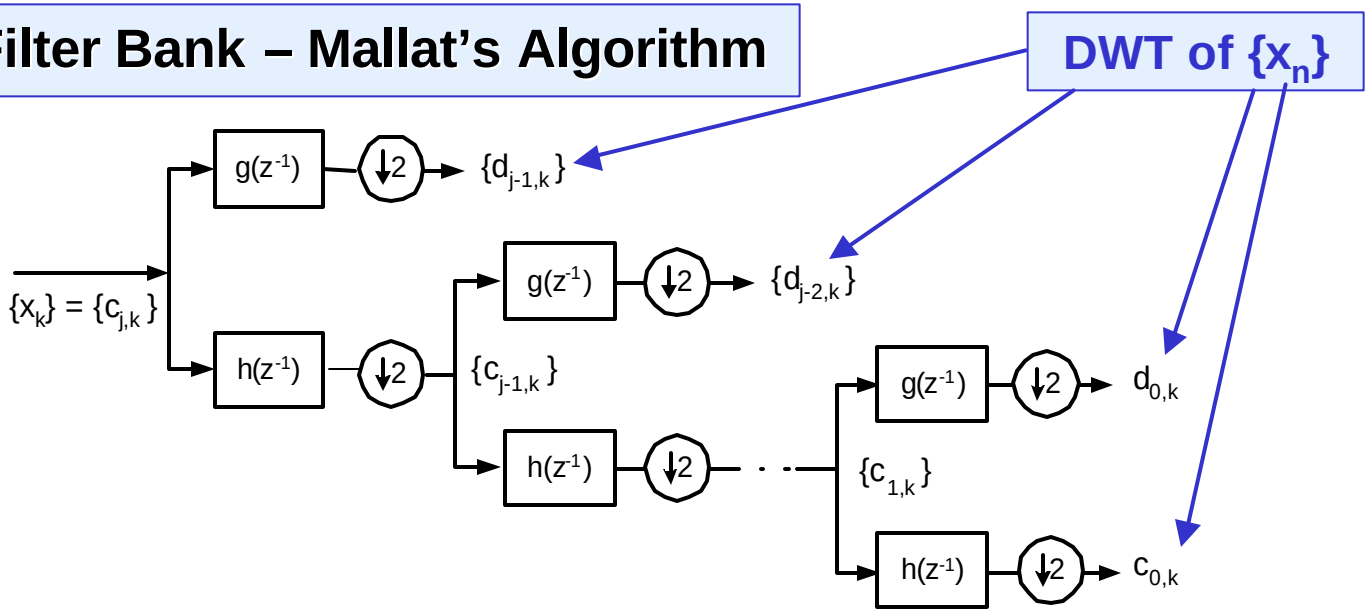
DWT: Rules and Algorithms

Filter Bank – Mallat's Algorithm



Mallat Rule

Extension matrix

$$\mathrm{DWT}_n(\mathbf{h},\mathbf{g}) \to \begin{bmatrix} \mathrm{DWT}_{n/2}(\mathbf{h},\mathbf{g}) \to \mathrm{I}_{n/2} \otimes \mathbf{2} \otimes F_{n/2}(\mathbf{h}) \to \\ \to \mathrm{I}_{n/2} \otimes \mathbf{2} \otimes F_{n/2}(\mathbf{g}) \end{bmatrix} \mathbf{E}^*_{n,l,r}$$

Fused Mallat Rule

$$\mathrm{DWT}_n(\mathbf{h},\mathbf{g}) \to \begin{bmatrix} \mathrm{DWT}_{n/2}(\mathbf{h},\mathbf{g}) \to \mathrm{I}_{n/2} \otimes H_n(\mathbf{h},\mathbf{g}) \end{bmatrix} \mathbf{E}^*_{n,l,r}$$

- **Redundant computations removed**
- **Filter structure lost**

$$H_n(\mathbf{h},\mathbf{g}) \to \begin{bmatrix} \mathrm{I}_{n/2} \otimes_{k\to 2} \mathbf{h}^T \\ \mathrm{I}_{n/2} \otimes_{k\to 2} \mathbf{g}^T \end{bmatrix}$$

single-level DWT

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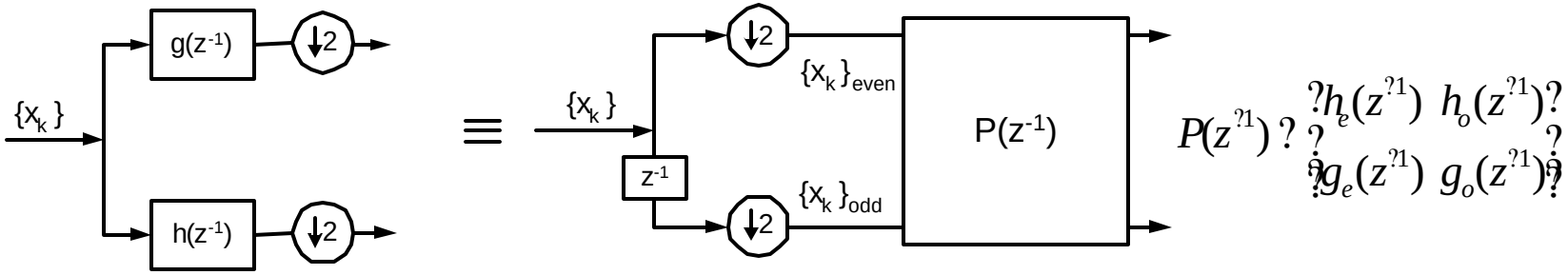
Polyphase Representation



$$P(z^{-1}) = \begin{bmatrix} h_e(z^{-1}) & h_o(z^{-1}) \\ g_e(z^{-1}) & g_o(z^{-1}) \end{bmatrix}$$

Polyphase Rule

Gateway to frequency domain computation

$$H_n(\mathbf{h},\mathbf{g}) = \begin{bmatrix} F_{n/2}^T(\mathbf{h}_o) & F_{n/2}^T(\mathbf{h}_e) \\ F_{n/2}^T(\mathbf{g}_o) & F_{n/2}^T(\mathbf{g}_e) \end{bmatrix} L_2^{n,k,2}$$

Lifting Scheme

Primal (Predict) **Dual (Update)**

Lifting steps

$$\begin{bmatrix} h_e(z^{-1}) & h_o(z^{-1}) \\ g_e(z^{-1}) & g_o(z^{-1}) \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & 1/K \end{bmatrix} \begin{bmatrix} 1 & s_i(z^{-1}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_i(z^{-1}) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & s_0(z^{-1}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_0(z^{-1}) & 1 \end{bmatrix}$$

- **Asymptotically reduce cost by 50%**
- **In-place computations**

Lifting Rule

$$H_n(\mathbf{h},\mathbf{g}) = \begin{bmatrix} K I_{n/2} & 0 \\ 0 & 1/K I_{n/2} \end{bmatrix} \begin{bmatrix} I_{n/2} & F_{n/2}^T(\mathbf{s}_i) \\ 0 & I_{n/2} \end{bmatrix} \cdots \begin{bmatrix} I_{n/2} & 0 \\ F_{n/2}^T(\mathbf{t}_o) & I_{n/2} \end{bmatrix} L_2^{n,k,2}$$

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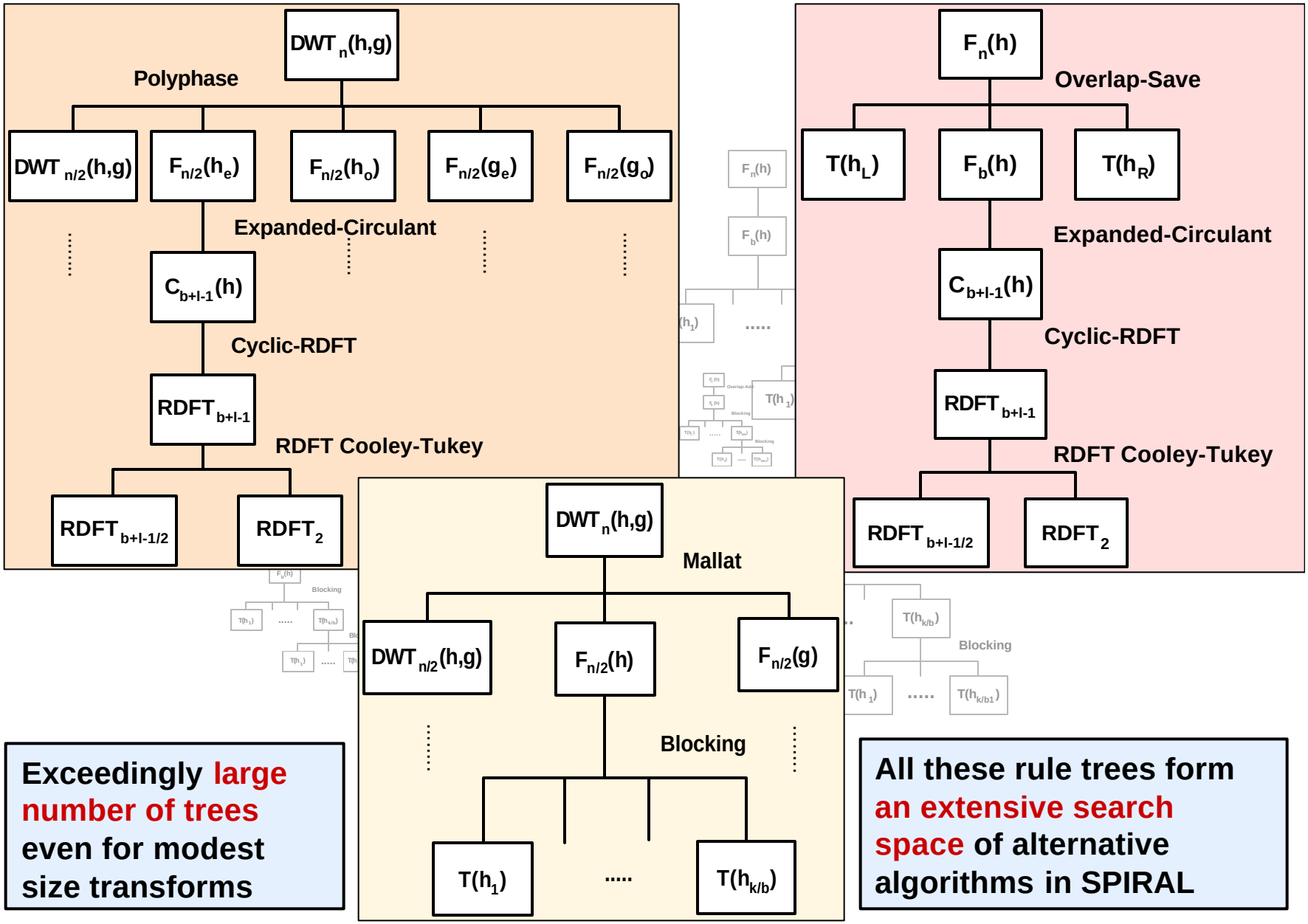
Examples: FIR Filter and DWT Rule Trees



Exceedingly large number of trees even for modest size transforms

All these rule trees form an extensive search space of alternative algorithms in SPIRAL

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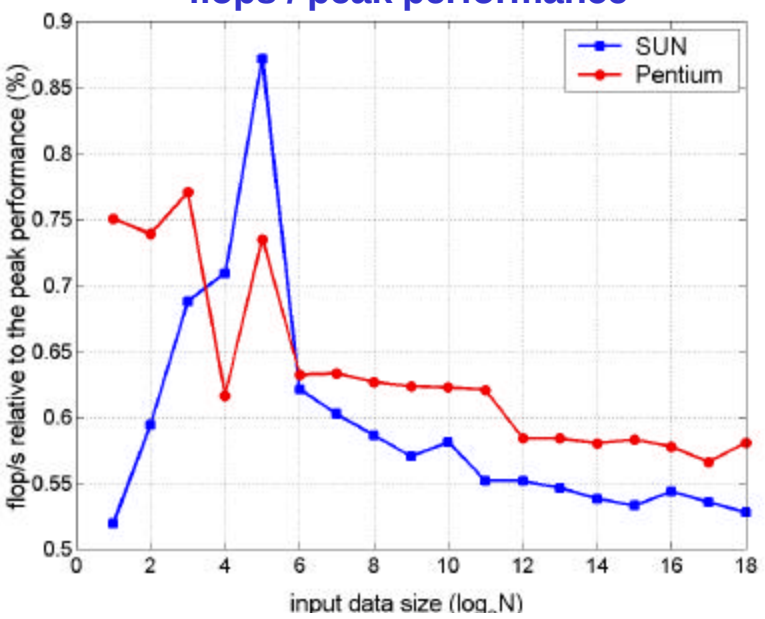
Time-Domain Methods

FIR Filter

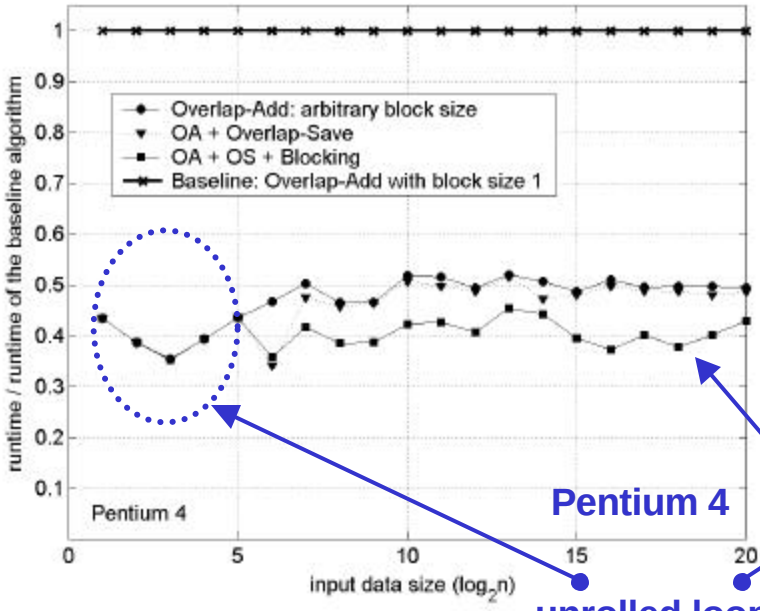
- overlap-add, overlap-save, and blocking
- all methods have the same cost
- baseline method = overlap-add with *b*=1
- sanity check – percentage of the peak performance

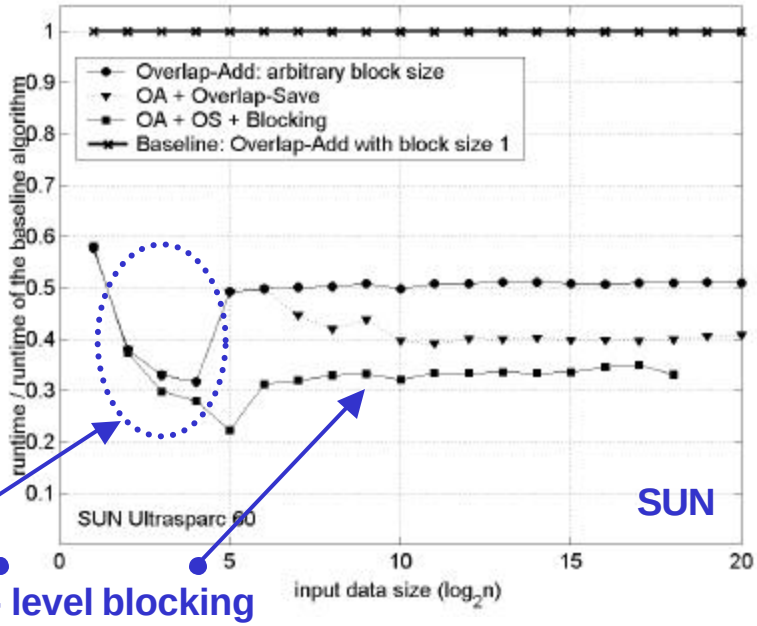
60-70% improvement over baseline
Same arithmetic cost

flops / peak performance



comparison of different time-domain methods





Pentium 4

SUN

unrolled loops

1- level blocking
blocks size 2

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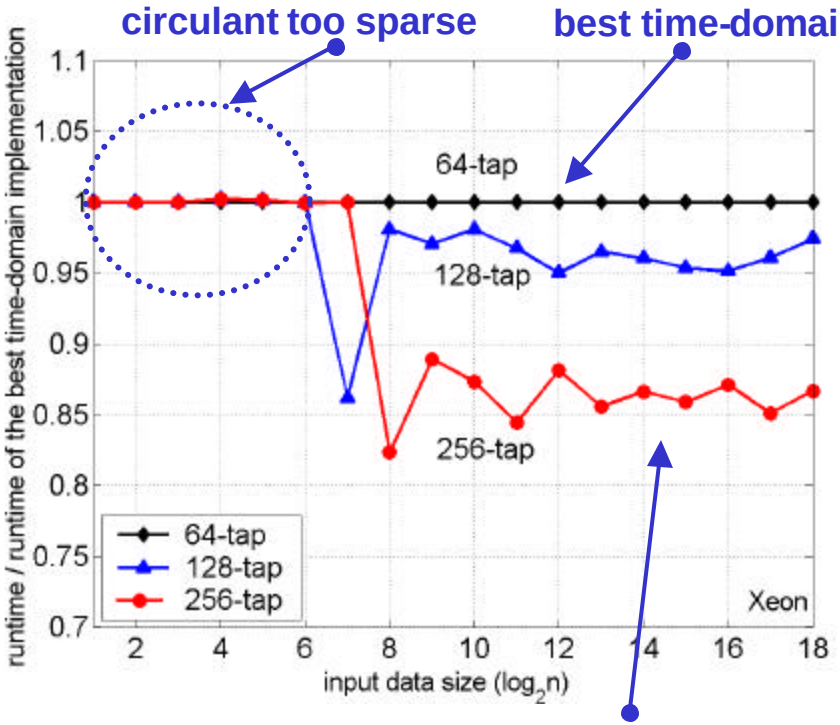
Frequency Domain vs. Time Domain

FIR Filter

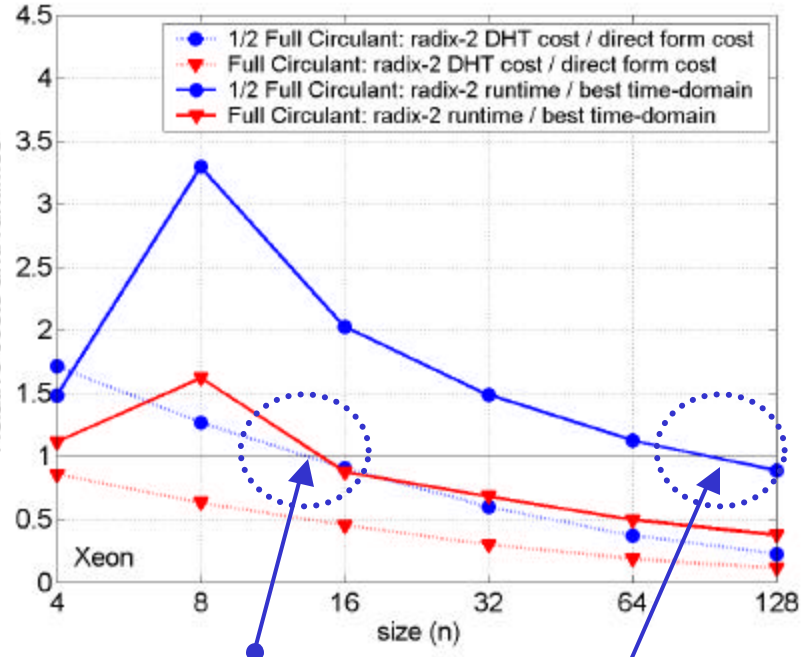
Frequency domain – potentially lower cost
Time domain – beter data access patterns



FD methods better on large "fat" circulants
Filter lengths > 64

Circulant

"half–dense" circulants emerge in frequency domain methods



arithmetic cost cross-over

runtime cross-over

Considerable discrepancy between arithmetic costs and actual runtimes

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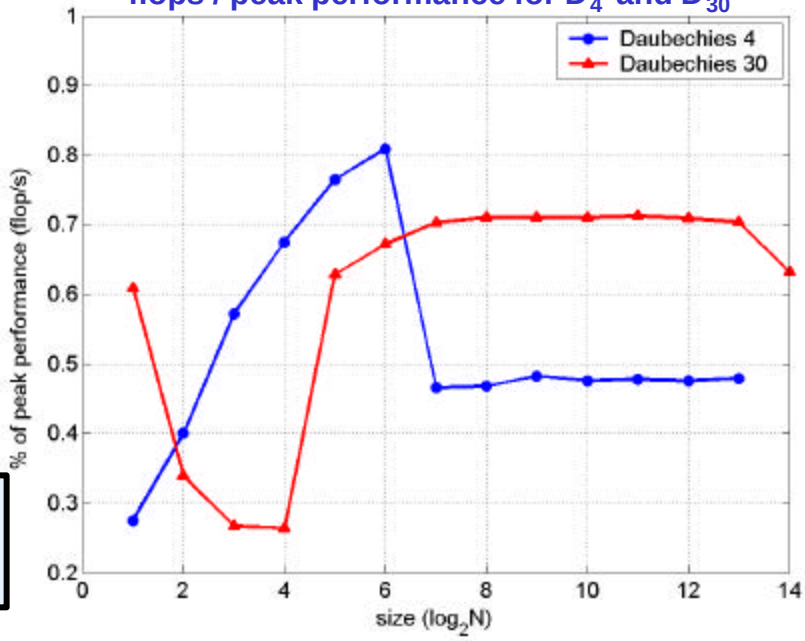
Daubechies D_4 DWT

Short Wavelet

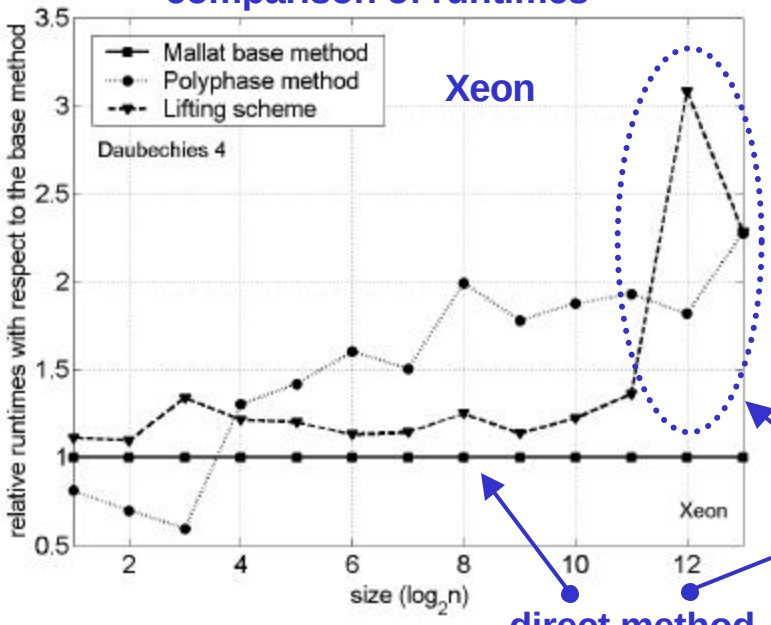
- single-level DWT using D_4
- 3 different classes of rule trees based on the initial rule
  - **Fused Mallat Rule** (direct method)
  - **Polyphase Rule** (frequency domain)
  - **Lifting Rule** (lifting scheme)

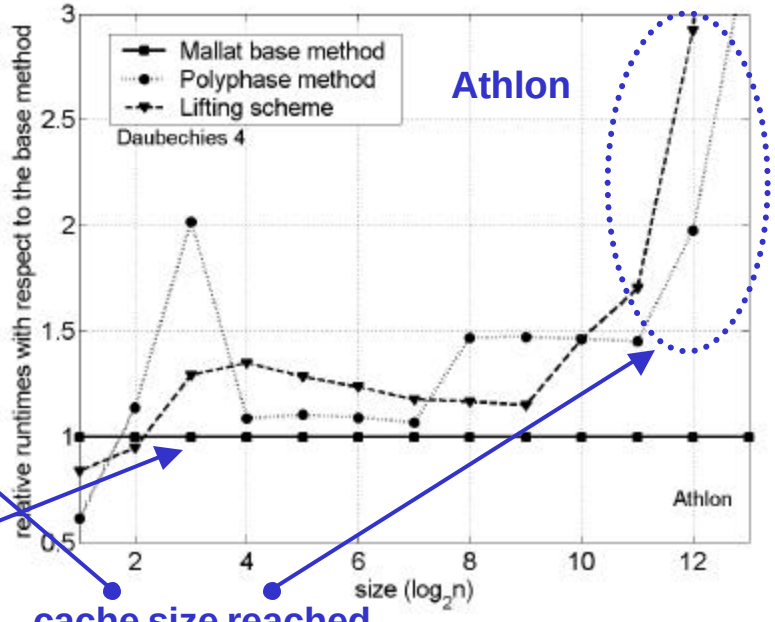
Frequency domain trees not found
Lifting scheme – 30% lower cost – slower

flops / peak performance for D_4 and D_{30}



comparison of runtimes





direct method

cache size reached

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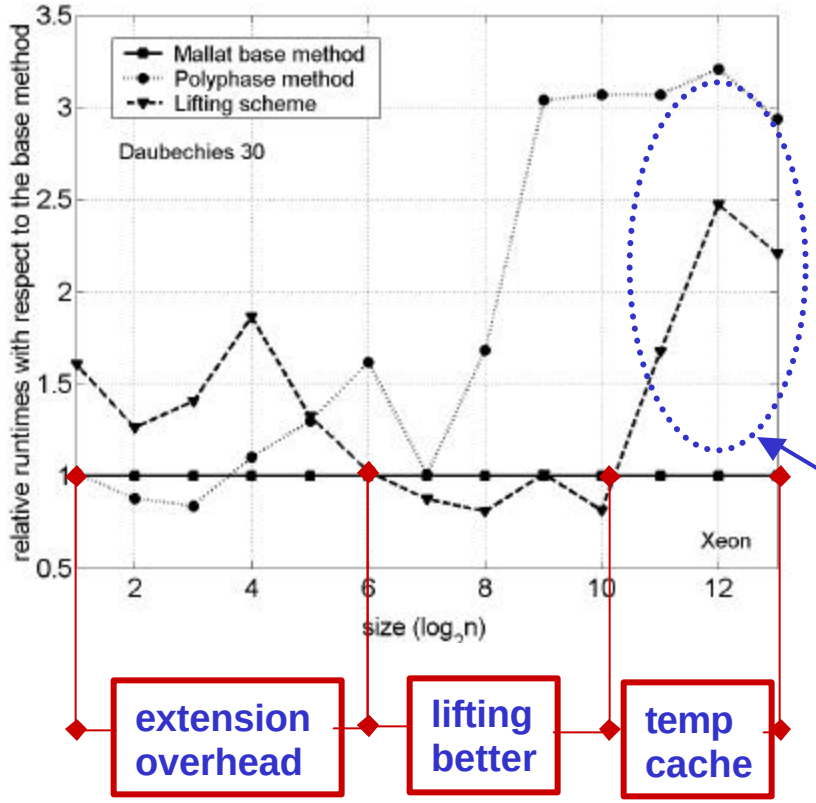
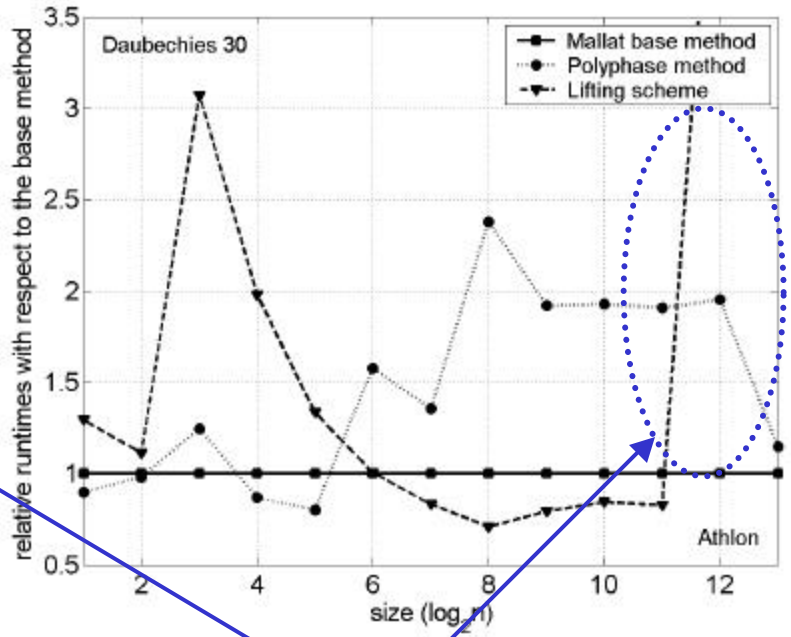
Daubechies D_{30} DWT Experiment

Long Wavelet

Xeon



Athlon



Different methods are found for different size ranges
Best implementations vary from machine to machine

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Summary

- Fast automatically generated code for FIR filters and DWT
  - Significant contribution – no existing solutions
- Preliminary results
  - Arithmetic cost not a reliable predictor of performance
  - Best implementation not obvious for a human programmer
  - Best implementations vary over different computer platforms

What lies ahead?

- Further algorithmic and implementation optimizations
  - Fusion of the extension, exploiting in-place structure, scheduling,..
- Entire class of wavelets and extension methods
  - Orthogonal, biorthogonal, frames,…
  - Periodic, symmetric, polynomial extension, zero-padding
- Extend to other wavelet transforms
  - 2D-DWT, M-band, wavelet packet transform
- Utilize special instruction set extensions
  - Single Instruction Multiple Data (SIMD)
  - Fused Multiply-Add (FMA)